Simplicial Homology

I: Simplices

II: Simplicial Complexes

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IV: Homology: detecting "nice" holes

Reference: J.R. Munkres, "Elements of Algebraic Topology", Perseus Publishing, 1984, ISBN 0-201-62728-0.

I: Simplices

Definition:

Let $\{a_0, a_1, ..., a_k\}$ be points in $\mathbb{R}^n$. This set is said to be geometrically independent if the vectors

$$a_1 - a_0, \quad a_2 - a_0, \quad ..., \quad a_k - a_0$$

are linearly independent (as in linear algebra).

Remark: We impose that singletons be considered geometrically independent.

Definition:

Let $\{a_0, a_1, ..., a_k\}$ be a geometrically independent set in $\mathbb{R}^n$. A k-simplex σ spanned by these points is the set of points $x \in \mathbb{R}^n$ such that

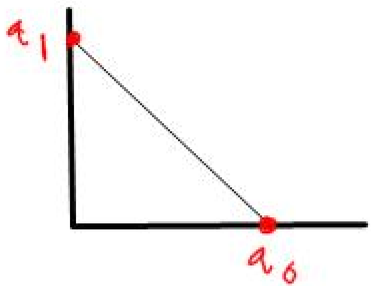
$$x = \sum_{i=0}^{k} t_i a_i \quad \text{where} \quad \sum_{i=0}^{k} t_i = 1$$

and $t_i \geq 0$ for all i.

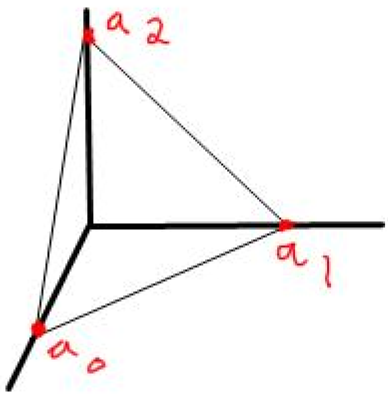
Remark: A k-simplex spanned by $a_0, a_1, ..., a_k$ is the **convex hull** of these points.

Examples:

0 - simplex



1 - simplex



2- simplex

Let σ be a k-simplex spanned by $\{a_0, a_1, ..., a_k\}$.

Definition:

1. The points $a_0, a_1, ..., a_k$ are called the **vertices** of σ.

2. The number k is the **dimension** of σ.

3. Any simplex spanned by a subset of $\{a_0, a_1, ..., a_k\}$ is called a **face** of σ.

4. The face spanned by $\{a_0, a_1, ..., a_k\} - \{a_i\}$ for some i is called the **face opposite** to a_i.

II: Simplicial Complexes

Definition:

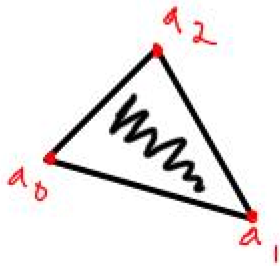
A simplicial complex K in $\mathbb{R}^n$ is a collection of simplices in $\mathbb{R}^n$ (of possibly varying dimensions) such that

1. Every face of a simplex of K is in K.

2. The intersection of any two simplices of K is a face of each.

Examples:

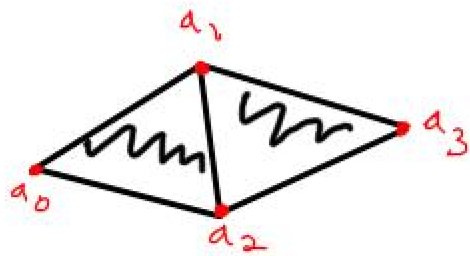
K1:



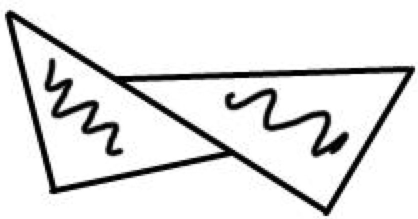
this 2-simplex together with all its faces is a simplicial complex.

$$\{\{a_0, a_1, a_2\}, \{a_0, a_1\}, \{a_0, a_2\}, \{a_1, a_2\}, \{a_0\}, \{a_1\}, \{a_2\}\}$$

K2:

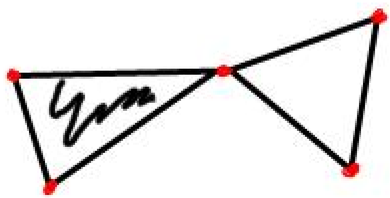


K3:



Is not a simplicial complex

K4:



Is a simplicial complex

Definition:

If L is a subcollection of K that contains all faces of its elements, then L is a simplicial complex. It is called a subcomplex of K

Remark: Given a simplicial complex K, the collection of all simplices of K of dimension at most p is called the p-skeleton of K and is denoted $K^{(p)}$.

e.g. $K^{(0)}$ is the set of vertices of K.

Definition:

If there exists an integer N such that

$$K^{(N-1)} \neq K \quad \text{and} \quad K^{(\geq N)} = K,$$

then K is said to have dimension N. Otherwise it is said to have infinite dimension.

Remark: A simplicial complex K is said to be finite if $K^{(0)}$ is finite.

Topology:

Let K be a simplicial complex in $\mathbb{R}^n$ and consider the set

$$|K| \quad = \quad \bigcup_{\sigma \in K} \sigma.$$

There are two natural ways of putting a topology on $|K|$:

1) $|K|$ being a subset of $\mathbb{R}^n$, the subspace topology would be a natural choice.

2) Giving each simplex σ of K its natural topology as a subspace of $\mathbb{R}^n$, declare a subset A of $|K|$ to be **closed** if

$$A \cap \sigma$$

is closed in σ for all $\sigma \in K$.

Remarks:

1. The set $|K|$ together with the second topology is the **realization** of K.

2. In general the second topology is finer (larger) than the first one.

3. These two topologies coincide for finite simplicial complexes.

Example:

Consider the following simplicial complex of the real line:

$$K \quad = \quad \{[n, n+1]\}_{n\neq 0} \quad \bigcup \quad \left\{[\frac{1}{n+1}, \frac{1}{n}]\right\}_{n\in\mathbb{Z}^+} .$$

Clearly, as sets, $|K| = \mathbb{R}$, but **NOT** as topological spaces,

e.g., the set $\left\{\frac{1}{n}\right\}_{n\in\mathbb{Z}^+}$ is closed in $|K|$ but not in $\mathbb{R}$.

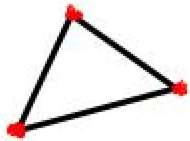
Definition:

A triangulation of a topological space X is a simplicial complex K together with a homeomorphism

$$|K| \longrightarrow X.$$

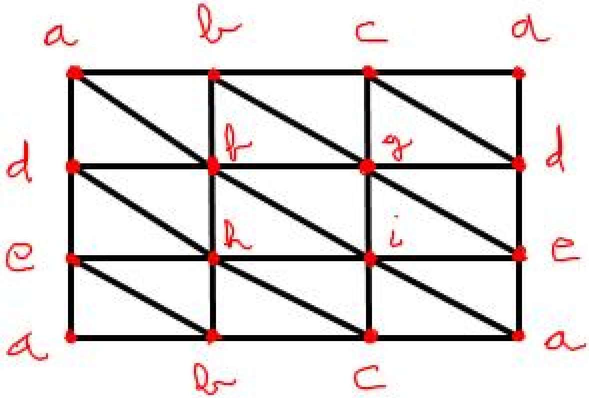
Examples:

K1:

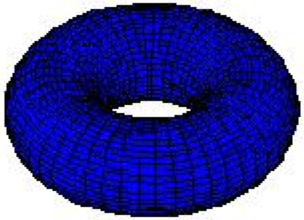


|K1| is homeomorphic to the circle

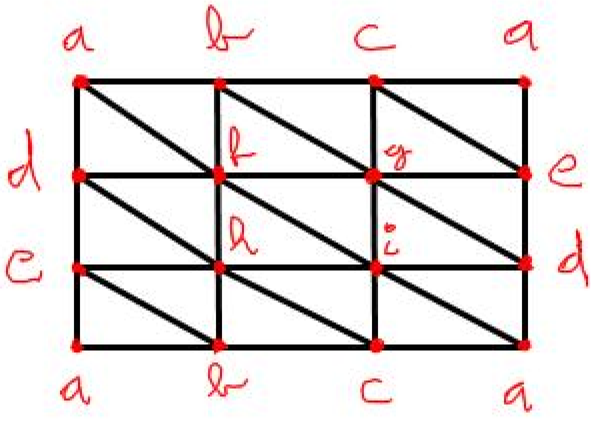
K2:



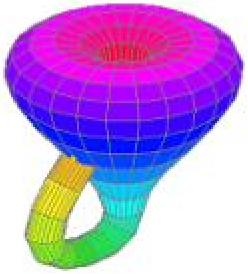
|K2| =



K3:



|K3| =



III: Fields versus Principal Ideal Domains (PID)

Review:

Let R be a commutative ring with unity 1.

Remark: The ring R is an integral domain if it has no zero divisors.

A. Fields

Let R be a field and V and W be two finite dimensional R-vector spaces. Consider an R-linear map

$$T : V \to W.$$

Theorem A: If $dim(V) = dim(W)$, then the following are equivalent

1. T is injective.
2. T is surjective.
3. T is an isomorphism.

Theorem B: The $Im(T)$ and $Coker(T)$ determines W, i.e.,

$$W \quad \cong \quad Im(T) \oplus Coker(T).$$

B. PIDs

Recall that a ring R is a PID if it is an integral domain and every ideal in R is principal, i.e., each ideal in R has a generating set consisting of a single element. Thus we have greatest common divisors (gcd's).

e.g., the integers: $\mathbb{Z}$.

Theorem C: If R is a field, then $R[x]$ is a PID.

Theorem D: If R is a PID and M is a free R-module, then any submodule N of M is free. Moreover, its rank is less than or equal to the rank of M.

Remarks:

1. When R is a PID, Theorem A is false in general, e.g.,

$$\phi : \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}$$

   is injective as a $\mathbb{Z}$-linear map but not onto!

2. Theorem B is also false when R is a PID, e.g., consider the same map ϕ as in the preceding example.

$$\mathbb{Z} \quad \ncong \quad 2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

C. Extensions

The last remark opens up a vast subject. Consider two R-modules A and C, and the following diagram

$$A \xrightarrow{i} ? \xrightarrow{p} C.$$

Question: How many different R-modules M (up to isomorphism) can we put in the middle of that diagram such that

1. the map i is injective;
2. the map p is surjective; and
3. $Im(i) = ker(p)$?

Answer: $Ext(C, A)$

Theorem E: For any abelian group A and positive integer m we have

$$Ext(\mathbb{Z}/m\mathbb{Z}, A) \quad \cong \quad A/mA.$$

e.g., $Ext(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, i.e., there are two possible extensions

$$\mathbb{Z} \quad \xrightarrow{\times 2} \quad \mathbb{Z} \quad \xrightarrow{p} \quad \mathbb{Z}/2\mathbb{Z}$$

and

$$\mathbb{Z} \quad \xrightarrow{i} \quad \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad \xrightarrow{p} \quad \mathbb{Z}/2\mathbb{Z}$$

IV: Homology: detecting "nice" holes

A: Ordered simplices

Let σ be a simplex. Two orderings of its vertex set are equivalent if they differ by an even permutation.

If $dim(\sigma) > 0$ then the orderings of the vertices of σ fall into two equivalence classes.

Each class is called an orientation of σ.

Definition:

An oriented simplex is a simplex σ together with an orientation of σ.

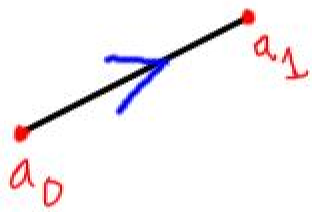
If $\{a_0, a_1, ..., a_p\}$ spans a p-simplex σ, then we shall use the symbol

$$[a_0, a_1, ..., a_p]$$

to denote the oriented simplex.

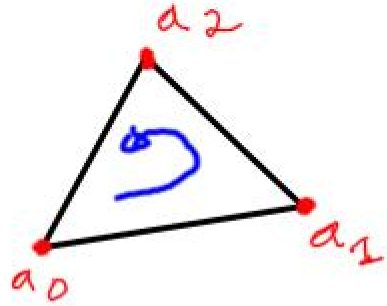
Remark: Clearly 0-simplices have only one orientation.

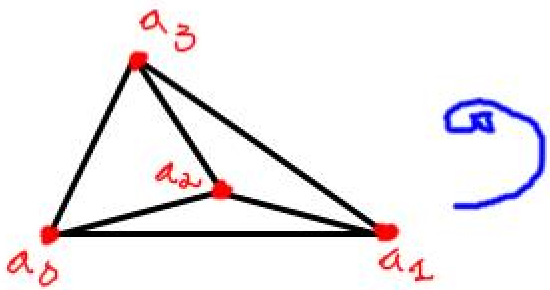
Examples:



1-oriented simplex

2-oriented simplex





3-oriented simplex

B: p-chains

Let K be a simplicial complex and G an abelian group.

Definition: A p-chain of K with coefficients in G is a function c_p from the oriented p-simplices of K to G that vanishes on all but finitely many p-simplices, such that

$$c_p(\sigma') = -c_p(\sigma)$$

whenever σ' and σ are opposite orientations of the same simplex.

The set of p-chains is denoted by $\boxed{C_p(K;G)}$. Moreover, it carries a natural abelian group structure, i.e, given $c_p, e_p \in C_p(K;G)$ we define

$$(c_p + e_p)(\sigma) = c_p(\sigma) + e_p(\sigma).$$

Remark: If $p < 0$ or $p > dim(K)$, then we set $C_p(K;G) = 0$.

Special case: $G = \mathbb{Z}$

If σ is an oriented simplex, there is an associated elementary chain c such that

1. $c(\sigma) = 1$;

2. $c(\sigma') = -1$ if σ' is the opposite orientation of σ; and

3. $c(\tau) = 0$ for all other oriented simplices τ.

Remark: By abuse of notation we will use the symbol σ to represent the associated elementary chain c, i.e.,

$$\sigma' = -\sigma.$$

Theorem F:

$C_p(K;\mathbb{Z})$ is a free abelian group; a basis can be obtained by orienting each p-simplex and using the corresponding elementary chains as a basis.

Definition:

We now define a homomorphism

$$\partial_p : C_p(K;\mathbb{Z}) \to C_{p-1}(K;\mathbb{Z})$$

called the boundary operator.

Let $p > 0$ and $\sigma = [v_0, ..., v_p]$ be an oriented simplex. Then

$$\partial_p \sigma \quad = \quad \sum_{i=0}^{p} \quad (-1)^i [v_0, ..., \hat{v}_i, ..., v_p]$$

where $\hat{v}_i$ means that the vertex v_i as been omitted.

Remarks:

1. It is routine to check that ∂_p is well defined.

2. You then extend linearly (using Theorem F) to the full $C_p(K;\mathbb{Z})$.

3. The boundary operators $\partial_{\leq 0}$ are set to 0 since $C_{p<0}(K;\mathbb{Z}) = 0$.

Examples:

1. 1-simplex: $\partial_1[v_0, v_1] = v_1 - v_0.$

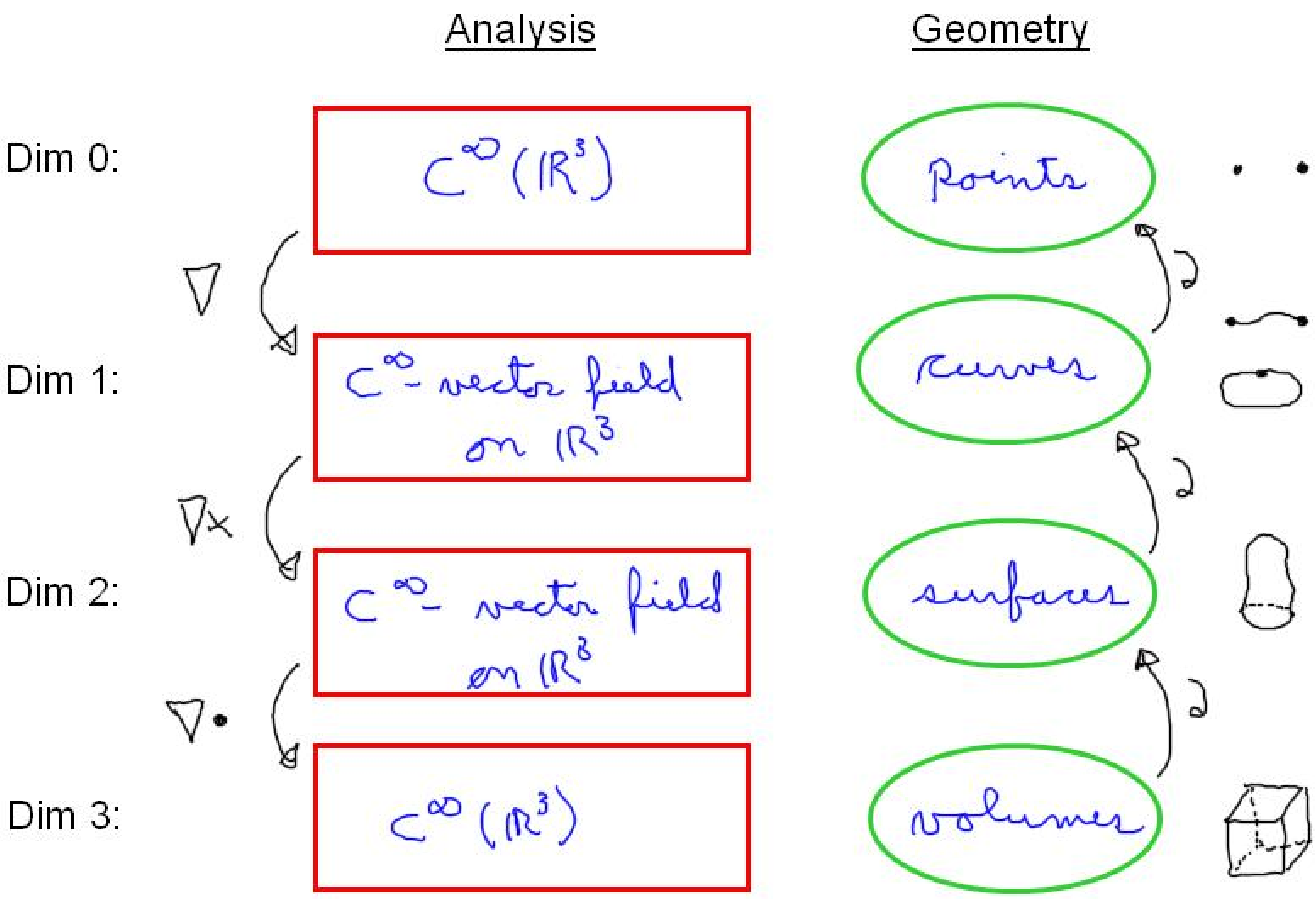
2. 2-simplex: $\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$

3. 3-simplex: $\partial_3[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$

Remark: Notice that $\partial_1 \circ \partial_2 = 0$.

Theorem G: $\partial_{p-1} \circ \partial_p \equiv 0.$

Analogy with calculus

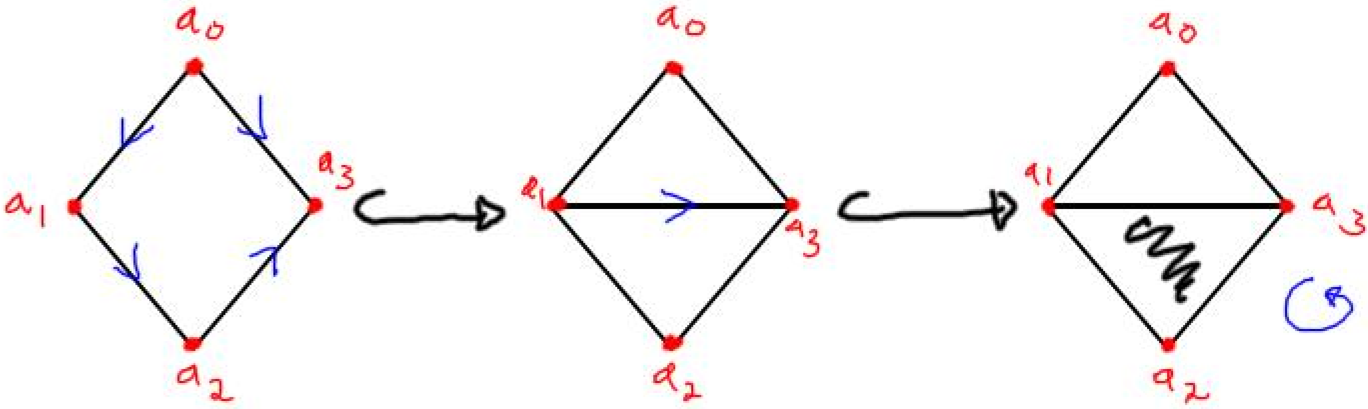


$$\nabla \times \nabla f = 0$$

$$\nabla \cdot \nabla \times F = 0$$

$$\partial^2 = \emptyset$$

Detecting Holes: Simplicial Homology



K1

K2

K3

$$C = [a_0 a_1] + [a_1 a_2] + [a_2 a_3] - [a_0 a_3]$$

$$\sigma_1 = [a_0 a_1] + [a_1 a_3] - [a_0 a_3]$$

$$\sigma_2 = [a_1 a_2] + [a_2 a_3] - [a_1 a_3]$$

$$[a_1 a_2 a_3]$$

Some computations:

K_1**:** Notice that

$$\partial_1(c) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) - (a_3 - a_0) \quad = \quad 0.$$

At first sight, $\boxed{ker(\partial)}$ seems to measure holes.

K_2**:** It seems that K_2 has three holes, since $\partial_1(v_1) = \partial_1(v_2) = \partial_1(c) = 0$. But clearly

$$c = v_1 + v_2,$$

i.e., in K_1, $dim(ker(\partial_1)) = 1$, and in K_2, $dim(ker(\partial_1)) = 2$.

K_3: c, v_1, and v_2 are still in K_3, but v_2 is no longer representing a hole!

How do we get rid of it?

Consider the 2-simplex $[a_1, a_2, a_3]$.

Then

$$\partial_2[a_1, a_2, a_3] = [a_2, a_3] - [a_1, a_3] + [a_1, a_2] = v_2.$$

i.e., $v_2 \in Im(\partial_2)$.

These observations together with Theorem G ($\partial^2 = 0$), suggest the following.

Definition: Let

1. $Z_k = ker(\partial_k)$, which we call k-cycles; and

2. $B_k = Im(\partial_{k+1})$, which we call k-boundaries.

Remark: Theorem G implies that $B_k \subset Z_k$.

Then the k^{th}-homology group of K is

$$H_k(K;\mathbb{Z}) \quad = \quad Z_k/B_k.$$

Summary:

1. $H_1(K_1) \cong \mathbb{Z}$;

2. $H_1(K_2) \cong \mathbb{Z} \oplus \mathbb{Z}$; and

3. $H_1(K_3) \cong \mathbb{Z}$. The cycles c and v_1 in K_3 actually represent the same homology class, i.e., they differ by a boundary namely,

$$v_1 \quad = \quad c \quad - \quad \partial[a_1, a_2, a_3].$$

The effect of changing coefficients

Let T denote the torus and K the Klein bottle.

1. One can show that over $\mathbb{Z}$

$$H_1(T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H_2(T;\mathbb{Z}) \cong \mathbb{Z},$$

while

$$H_1(K;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad H_2(K;\mathbb{Z}) = 0.$$

2. If one considers $\mathbb{Z}/2\mathbb{Z}$-coefficients, then

$$H_1(T;\mathbb{Z}/2\mathbb{Z}) \quad \cong \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad \cong \quad H_1(K;\mathbb{Z}/2\mathbb{Z})$$

and

$$H_2(T;\mathbb{Z}/2\mathbb{Z}) \quad \cong \quad \mathbb{Z}/2\mathbb{Z} \quad \cong \quad H_2(K;\mathbb{Z}/2\mathbb{Z})$$