Jump-adapted discretization schemes for Lévy-driven SDEs

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Weak approximation of Lévy-driven SDEs

We are interested in numerical evaluation of

$$E[f(X_1)], \quad \text{where} \quad X_t = X_0 + \int_0^t h(X_{s-})dZ_s, \quad X \in \mathbb{R}^n$$

where $Z \in \mathbb{R}^d$ is a pure-jump Lévy process:

$$Z_t = \gamma t + \int_0^t \int_{|y|\leq 1} y\widehat{N}(dy, ds) + \int_0^t \int_{|y|>1} yN(dy, ds)$$

Discretization with constant time step

The Euler scheme with constant time step

$$\hat{X}^n_{\frac{i+1}{n}} = \hat{X}^n_{\frac{i}{n}} + h(\hat{X}^n_{\frac{i}{n}})(Z_{\frac{i+1}{n}} - Z_{\frac{i}{n}})$$

has the convergence rate (Protter and Talay '97) $|E[f(X_1)] - E[f(\hat{X}^n_1)]| \leq \frac{C}{n}$ but suffers from two difficulties

- The increments of Z cannot usually be simulated in closed form;
- A large jump in Z between two discretization dates may lead to a large discretization error.

Jump-adapted discretization

A natural idea to solve both problems, due to Rubenthaler '03, is

- Approximate Z with a compound Poisson process

$$Z_t^\varepsilon := \gamma_\varepsilon t + \int_0^t \int_{|y|>\varepsilon} yN(dy, ds), \quad \gamma_\varepsilon = \gamma - \int_{\varepsilon<|y|\le 1} y\nu(dy).$$

- Apply the Euler scheme at every jump time of Z^ε.

The convergence rate may be computed in terms of *expected* number of discretization dates, proportional to $\lambda_\varepsilon = \int_{|y|\ge\varepsilon} \nu(dy)$.

This rate may range from very good to very bad (for Z of infinite variation) because

- The variance of small jumps may go to zero very slowly;
- The drift γ_ε may explode as $\varepsilon \to 0$.

Taking into account the structure of Z

- View the Lévy process between the jumps of Z^ε as a deterministic ODE perturbed by noise (small jumps).
- Approximate small jumps by Brownian motion (Asmussen and Rosinski '01):

$$\left(\int_0^t \int_{|z|\leq\varepsilon} y\hat{N}(dy, ds)\right)_{0\leq t\leq 1} \approx (W_t^\varepsilon)_{0\leq t\leq 1}$$

  where W^ε is a d-dimensional BM with covariance

$$\Sigma_{ij}^\varepsilon = \int_{|y|\leq\varepsilon} y_i y_j \nu(dy).$$

- Solve the deterministic ODE explicitly, or with a higher-order scheme, which is easy to construct.
- Expand the solution to the SDE around the explicit solution of the ODE

Constructing the approximating process

- Start by replacing the small jumps of Z with a Brownian motion:

$$d\bar{X}_t = h(\bar{X}_{t-})\{\gamma_\varepsilon dt + dW^\varepsilon_t + dZ^\varepsilon_t\},$$

where W^ε is a d-dimensional Brownian motion with covariance matrix Σ^ε. This process can also be written as

$$\bar{X}(t) = \bar{X}(\eta_t) + \int_{\eta_t}^t h\left(\bar{X}(s)\right) dW^\varepsilon(s) + \int_{\eta_t}^t h\left(\bar{X}(s)\right) \gamma_\varepsilon ds,$$
$$\bar{X}(T^\varepsilon_i) = \bar{X}(T^\varepsilon_i-) + h(\bar{X}(T^\varepsilon_i-))\Delta Z(T^\varepsilon_i),$$

Expanding the solution

- Consider a family of processes $(Y^\alpha)_{0\le\alpha\le 1}$ defined by

$$Y^\alpha(t) = \bar{X}(\eta_t) + \alpha \int_{\eta_t}^t h(Y^\alpha(s))\, dW^\varepsilon(s) + \int_{\eta_t}^t h(Y^\alpha(s))\,\gamma_\varepsilon ds$$

- Our idea is to replace the process $\bar{X} := Y^1$ with its first-order Taylor approximation:

$$\bar{X}(t) \approx Y^0(t) + \frac{\partial}{\partial\alpha} Y^\alpha(t)|_{\alpha=0}.$$

The approximation

The new approximation $\tilde{X}$ is defined by

$$\tilde{X}(t) = Y_0(t) + Y_1(t), \quad t > \eta_t,$$
$$\tilde{X}(T_i^\varepsilon) = \tilde{X}(T_i^\varepsilon -) + h(\tilde{X}(T_i^\varepsilon -))\Delta Z(T_i^\varepsilon),$$
$$Y_0(t) = \tilde{X}(\eta_t) + \int_{\eta_t}^t h(Y_0(t))\gamma_\varepsilon ds$$
$$Y_1(t) = \int_{\eta_t}^t \frac{\partial h}{\partial x_i}(Y_0(s))\, Y_1^i(s)\gamma_\varepsilon ds + \int_{\eta_t}^t h(Y_0(s))\, dW^\varepsilon(s)$$

where we used the Einstein convention for summation over repeated indices.

Computing Y_0 and Y_1

- Y_0 is the solution of an ODE and can be computed, e.g., by a 4th order Runge-Kutta scheme.
- Conditionally on $(T_i^\varepsilon)_{i\geq 1}$, the random vector $Y_1(t)$ is Gaussian with mean zero, and we only need its terminal covariance.
- Its covariance matrix $\Omega(t)$ satisfies the linear equation

$$\Omega(t) = \int_{\eta_t}^{t} (\Omega(s)M(s) + M^\perp(s)\Omega^\perp(s) + N(s))ds$$

where $M^\perp$ denotes the transpose of the matrix M and

$$M_{ij}(t) = \frac{\partial h_{ik}(Y_0(t))}{\partial x_j}\gamma_\varepsilon^k \quad \text{and} \quad N(t) = h(Y_0(t))\Sigma^\varepsilon h^\perp(Y_0(t)).$$

- In one dimension, the solution is $\Omega(t) = \Sigma^\varepsilon h^2(Y_t^0)(t-\eta_t)$.

The convergence rate

($\mathbf{H_n}$) $f \in C^n$, $h \in C^n$ $f^{(k)}$ and $h^{(k)}$ are bounded for $1 \leq k \leq n$ and $\int z^{2n}\nu(dz) < \infty$.

($\mathbf{H'_n}$) $f \in C^n$, $h \in C^n$, $h^{(k)}$ are bounded for $1 \leq k \leq n$, $f^{(k)}$ have at most polynomial growth for $1 \leq k \leq n$ and $\int |z|^k \nu(dz) < \infty$ for all $k \geq 1$.

- Assume ($\mathbf{H_3}$) or ($\mathbf{H'_3}$). Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C \left(\frac{\|\Sigma^\varepsilon\|}{\lambda_\varepsilon}(\|\Sigma^\varepsilon\| + |\gamma_\varepsilon|) + \int_{|y|\leq\varepsilon} |y|^3 \nu(dy) \right)$$

- Assume ($\mathbf{H_4}$) or ($\mathbf{H'_4}$) and $\nu(dy) = (1 + \xi(y))\nu_0(dy)$, where ν_0 is a symmetric measure and $\xi(y) = O(y)$. Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C \left(\frac{\|\Sigma^\varepsilon\|}{\lambda_\varepsilon} + \int_{|y|\leq\varepsilon} |y|^4 \nu(dy) \right).$$

Worst-case bounds

For general Lévy measures,

$$|E[f(\hat{X}_1) - f(X_1)]| \leq o(\lambda_\varepsilon^{-\frac{1}{2}}),$$

and in the locally symmetric case,

$$|E[f(\hat{X}_1) - f(X_1)]| \leq o(\lambda_\varepsilon^{-1}).$$

- For all known examples, the convergence rates are better.

Stable-like behavior and other examples

- Assume that ν has a density $\nu(z) = \frac{g(z)}{|z|^{1+\alpha}}$ with g bounded near zero. Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq O(\lambda_\varepsilon^{(1-\frac{3}{\alpha})\vee(-\frac{2}{\alpha})}),$$

and if the Lévy measure is symmetric near zero (CGMY),

$$|E[f(\hat{X}_1) - f(X_1)]| \leq O(\lambda_\varepsilon^{-\frac{2}{\alpha}}).$$

- The NIG process has a symmetric stable-like Lévy measure with $\alpha = 1$

$$\Rightarrow \quad |E[f(\hat{X}_1) - f(X_1)]| \leq O(1/\lambda_\varepsilon^2)$$

- In the VG model, the convergence is exponential:

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C\frac{e^{-2\lambda_\varepsilon}}{\lambda_\varepsilon}.$$

Libor market models

- The Libor market model (general case of BGM model) describes joint arbitrage-free dynamics of a set of forward interest rates.
- Libor market models with jumps were considered by Jamshidian '99, Glasserman and Kou '03, Eberlein and Özkan '05, Papapantoleon and Skovmand '10 and others.
- Let $T_i = T_1 + (i-1)\delta$, $i = 1, \dots, n+1$ be a set of dates called *tenor* dates. The *Libor* rate L^i_t is the forward rate defined at t for the period $[T_i, T_{i+1}]$:

$$L^i_t = \frac{1}{\delta}\left(\frac{B_t(T_i)}{B_t(T_{i+1})} - 1\right),$$

where $B_t(T)$ is the price at t of a zero-coupon bond with maturity T.

Introducing jumps

Following Jamshidian '99, an arbitrage-free dynamics of n forward Libors $L_t^1, \dots, L_t^n$ can be constructed via the multi-dimensional SDE

$$\frac{dL_t^i}{L_{t-}^i} = \sigma^i(t) dZ_t - \int_{\mathbb{R}^d} \sigma^i(t) z \left[\prod_{j=i+1}^{n} \left(1 + \frac{\delta L_t^j \sigma^j(t) z}{1 + \delta L_t^j} \right) - 1 \right] \nu(dz) dt,$$

where Z is a d-dimensional martingale pure jump Lévy process with Lévy measure ν under the terminal measure Q and $\sigma^i(t)$ are deterministic volatility functions.

Terminal measure: martingale measure for which the last zero-coupon bond $B_t(T_{n+1})$ is the *numéraire*.

Technicalities: computing the coefficients

We are interested in computing the quantities of the form

$$C^i = \int_{\mathbb{R}^d} \sigma^i z \left[\prod_{j=i+1}^{n} (1 + a_j z) - 1 \right] \nu(dz)$$

Assume first $d = 1$. Then

$$C^i = \sum_{j=i+1}^{n} A^{ij} \sigma^i \int z^{j+1-i} \nu(dz), \qquad A^{ij} = \sum_{i+1 \leq k_{i+1} < \dots < k_j \leq n} a_{k_{i+1}} \dots a_{k_j}$$

The coefficients A^{ij} can be computed in polynomial time

$$A^{n-1,n} = a_n$$

$$A^{n-2,n} = a_n a_{n-1} \qquad A^{n-2,n-1} = a_n + a_{n-1}$$

$$A^{n-3,n} = a_n a_{n-1} a_{n-2} \qquad A^{n-3,n-1} = a_n a_{n-1} + a_{n-1} a_{n-2} + a_n a_{n-2}$$

$$A^{n-3,n-2} = a_n a_{n-1} a_{n-2}$$

Technicalities: computing the coefficients

Assume the number of factors is arbitrary (usually $d = 2$ or $d = 3$).

Introduce the symmetric tensor of moments of ν:

$$M^k_{i_1,\dots,i_k} = \int_{\mathbb{R}^d} z_{i_1} \dots z_{i_k} \nu(dz)$$

A^{ij} also becomes a symmetric tensor.

The number of independent coefficients of a symmetric tensor of order n on a vector space of dimension d is $\binom{d+n-1}{n}$.

Monte Carlo option pricing

- The price of any asset divided by $B_t(T_{n+1})$ is a martingale and in particular the price of an option which pays $H = h(L^1_{T_1}, \ldots, L^n_{T_1})$ at time T_1 (e.g. swaption) is given by

$$\pi_t(H) = B_t(T_{n+1}) E\left[\frac{h(L^1_{T_1}, \ldots, L^n_{T_1})}{B_{T_1}(T_{n+1})} \middle| \mathcal{F}_t \right]$$
$$= B_t(T_{n+1}) E\left[h(L^1_{T_1}, \ldots, L^n_{T_1}) \prod_{i=1}^n (1 + \delta L^i_{T_1}) \middle| \mathcal{F}_t \right].$$

- The price of any such option can therefore be computed by Monte Carlo using the Libor dynamics.

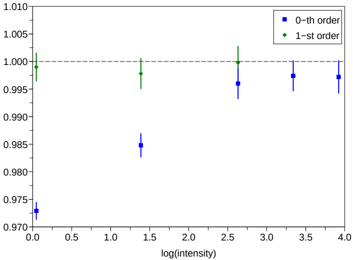
Numerical implementation

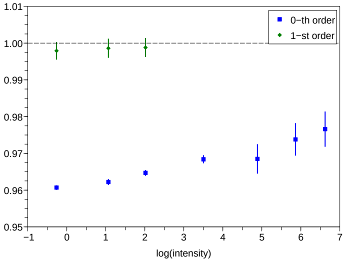
- We consider a Libor market model with tenor dates $\{5, 6, 7, 8, 9, 10\}$, a one-dimensional driving Lévy process and constant volatilities of all Libors ($\sigma^i(t) \equiv 1$).
- The initial values are fixed to 15% to emphasize the non-linear effects.
- The differential equations for $Y_0(t)$ and $\Omega(t)$ are solved simultaneously by fourth order Runge Kutta.
- The Lévy measure is

$$C\frac{e^{-\lambda_+ x}1_{x>0} + e^{-\lambda_- |x|}1_{x<0}}{|x|^{1+\alpha}}$$

with $\lambda_+ = 10$, $\lambda_- = 20$ and $\alpha = 0.5$, $C = 1.5$ (Case 1) or $\alpha = 1.8$, $C = 0.01$ (Case 2). Both cases correspond to annualized standard deviation of about 24%.

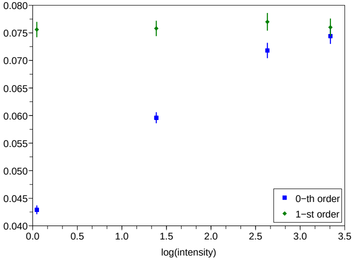
Sanity check: pricing a zero-coupon

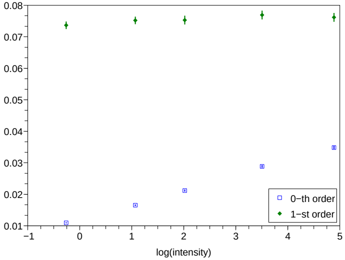




Ratio of estimated to theoretical zero coupon bond price in Case 1 (left) and Case 2 (right). The theoretical convergence rate is λ^{-4} (case 1) and $\lambda^{-1.11}$ (case 2). For comparison we also give results of the 0-order scheme (without Brownian approximation, in blue).

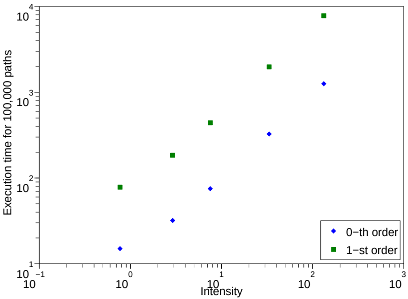
Pricing ATM receiver swaption





Estimated price of an ATM receiver swaption with maturity 5 years in Case 1 (left) and Case 2 (right).

Execution times



Execution times for the swaption example on a PIII PC without any code optimization.

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