On the Duality for Robust Utility Maximization with Unbounded Random Endowment

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Question

Utility Maximization with a Claim and Duality

- Utility Maximization with Claim:

$$\text{maximize} \quad E[U(\theta \cdot S_T + B)], \quad \text{among } \theta \in \Theta_{bb}.$$

  - S: locally bounded semimartingale,
  - $U : \mathbb{R} \to \mathbb{R}$: utility function,
  - Θ_{bb}: admissible integrands: $\theta \cdot S \geq \exists c$,
  - $B \in L^0$: payoff of a claim at maturity.
- Buyer's problem $\Rightarrow$ utility indifference price of B.
- Key Duality: letting $V(y) := \sup_x (U(x) - xy)$,

$$\sup_{\theta \in \Theta_{bb}} E[U(\theta \cdot S_T + B)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} E\left[V\left(\lambda \frac{dQ}{d\mathbb{P}}\right) + \lambda \frac{dQ}{d\mathbb{P}} B\right].$$

  - $\mathcal{M}_V = \{Q \in \mathcal{M}_{\text{loc}} : E[V(dQ/d\mathbb{P})] < \infty\}$.

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Many References!!

- No Claim ($B \equiv 0$ or constant):
  - Kramkov/Schachermayer 99, 03 ,
  - Schachermayer 01,
- Bounded Claim ($B \in L^\infty$):
  - Bellini/Frittelli 02,
  - Cvitanić/Schachermayer/Wang 01,
- Exponential Utility ($U(x) = -e^{-x}$):
  - Delbaen/Grandits/Rheinländer/Samperi/Schweizer/Stricker02,
  - Kabanov/Stricker 02,
- Unbounded B:
  - Owen/Žictović 09,
  - Biagini/Frittelli/Grasselli 10,
  - Owari 10,
- ....

Question: Robust Utility Maximization with a Claim

- What if we consider robust utility maximization?

$$\text{maximize} \quad \inf_{P \in \mathcal{P}} E^P[U(\theta \cdot S_T + B)], \quad \text{among } \theta \in \Theta_{bb}.$$

- $\mathcal{P}$: set of probabilities $\ll \mathbb{P} \Rightarrow$ Model Uncertainty.

Can we get a nice "duality"? How??

$$\sup_{\theta \in \Theta_{bb}} \inf_{P \in \mathcal{P}} E^P[U(\theta \cdot S_T + B)]$$

$$\overset{???}{=} \inf_{P \in \mathcal{P}} \inf_{\lambda > 0, Q \in \mathcal{M}} E^P\left[V\left(\lambda \frac{dQ}{dP}\right) + \lambda \frac{dQ}{dP} B\right]$$

- for a wide class of U and unbounded B.

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A Standard Way: Robust to Family of Subjectives

- Use minimax theorem:

$$\sup_\theta \inf_P E^P[U(\theta \cdot S_T + B)] \stackrel{?}{=} \inf_P \sup_\theta E^P[U(\theta \cdot S_T + B)]$$
$$\stackrel{?}{=} \inf_P \left(\inf_{\lambda, Q} E^P \left[V\left(\lambda \frac{dQ}{dP}\right) + \lambda \frac{dQ}{dP} B \right] \right)$$

- No problem if $B \equiv 0$ and $\mathrm{dom}(U) = \mathbb{R}_+$.
  - Schied/Wu 05, Schied 07.
  - Wittmüss 08 ($B \in L^\infty$ with singular term).
- Works also if $\sup_x U(x) < \infty \Rightarrow$ OK if $U(x) = -e^{-x}$.
- But...
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Conjugate of Robust Utility Functional with B

U: "nice" utility on $\mathbb{R}$ (Inada & Reas. Asymp. Elasticity).

- $u_{\mathcal{P},B}(X) := \inf_{P\in\mathcal{P}} E^P[U(X+B)]$: concave.
- $v_{\mathcal{P},B}(\nu) := \sup_{X\in L^\infty}(u_{\mathcal{P},B}(X) - \nu(X))$, $\nu \in ba$: conjugate.

Key Lemma

Under "suitable assumptions on B",

1. $u_{\mathcal{P},B}$ is continuous on L^∞,
2. $\forall \nu \in ba_+$,

$$v_{\mathcal{P},B}(\nu) = \begin{cases} V(\nu|\mathcal{P}) + \nu(B) & \text{if } \nu \ \sigma\text{-additive, } V(\nu|\mathcal{P}) < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

- $V(\nu|\mathcal{P}) := \inf_{P\in\mathcal{P}} V(\nu|P) := \inf_{P\in\mathcal{P}} \text{"}E^P[V(d\nu/dP)]\text{"}$

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Admitting Key Lemma

- $\mathcal{M}_V := \{Q \in \mathcal{M}_{\mathrm{loc}} : V(Q|\mathcal{P}) < \infty\}$. Assume: $\mathcal{M}_V^e \neq \emptyset$.
- $\mathcal{C} := \{X \in L^\infty : X \leq \theta \cdot S_T, \exists \theta \in \Theta_{bb}\}$: conv. cone, $\supset L^\infty_-$.
  - $\sup_{\theta \in \Theta_{bb}} u_{\mathcal{P},B}(\theta \cdot S_T)$ "=" $\sup_{X \in \mathcal{C}} u_{\mathcal{P},B}(X)$.
  - $\sup_{X \in \mathcal{C}} E^Q[X] = 0$ (resp. $= +\infty$) iff $Q \in \mathcal{M}_{loc}$ (resp. $\notin \mathcal{M}_{loc}$).

$$\sup_{X \in \mathcal{C}} u_{\mathcal{P},B}(X) = \sup_{X \in L^\infty} \left(u_{\mathcal{P},B}(X) - \delta_{\mathcal{C}}(X)\right) \overset{(1)}{=} \min_{\nu \in ba} \left(v_{\mathcal{P},B}(\nu) - \sup_{X \in \mathcal{C}} \nu(X)\right)$$

$$\overset{(2)}{=} \min_{\lambda > 0, Q \in \mathcal{M}_V} \left(V(\lambda Q|\mathcal{P}) + \lambda E^Q[B]\right).$$

1. Fenchel's theorem via the continuity of $u_{\mathcal{P},B}$.
2. Representation of $v_{\mathcal{P},B}$ (& "$L^\infty_- \subset \mathcal{C}$" + "RAE" + "$\mathcal{M}_V^e \neq \emptyset$").

$$\sup_{\theta \in \Theta_{bb}} \inf_{P \in \mathcal{P}} E^P[U(\theta \cdot S_T + B)] = \min_{\lambda > 0, Q \in \mathcal{M}_V} \left(V(\lambda Q|\mathcal{P}) + \lambda E^Q[B]\right).$$

Our Idea

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Robustification of Integral Functionals

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$: convex in $x \in \mathbb{R}$,
- $I_f(X) := E[f(\cdot, X)]$, $X \in L^\infty$: integral functional.

Rockafellar's Theorem

- $$\sup_{X \in L^\infty} (\nu(X) - I_f(X)) = E[f^*(\cdot, d\nu_r/d\mathbb{P})] + \sup_{X \in \mathrm{dom} I_f} \nu_s(X)$$
- If $f(\cdot, X) \in L^1$ $\forall X \in L^\infty$, I_f is continuous on L^∞, and
$$\sup_{X \in L^\infty} (\nu(X) - I_f(X)) = \begin{cases} E[f^*(\cdot, d\nu/d\mathbb{P})] & \text{if } \nu \text{ is } \sigma\text{-additive}, \\ +\infty & \text{otherwise.} \end{cases}$$

- What if $E[f(\cdot, X)] \Rightarrow I_{\mathcal{P},f}(X) := \sup_{P \in \mathcal{P}} E^P[f(\cdot, X)]$?
- $u_{\mathcal{P},B}(X) = -\sup_{P \in \mathcal{P}} I_{P,f}(-X)$, with $f(\omega, x) = -U(-x + B(\omega))$.
- Robust ver. of Rockafellar Th. $\Rightarrow$ Key Lemma $\Rightarrow$ duality.

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Robust Version of Rockafellar Theorem

Normal Convex Integrands and ω-wise Conjugate

- $(\Omega, \mathcal{F}, \mathbb{P})$: complete.

Normal Integrands

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ (not $\mathbb{R} \cup \{+\infty\}$): normal iff

- f is jointly measurable;
- $x \mapsto f(\omega, x)$: LSC, convex, proper.

- $\Rightarrow f^*(\cdot, y) := \sup_x (xy - f(\cdot, x))$ is also normal.
- $\tilde{f}(\omega, y, z) := (zf(\omega, \cdot))^*(y) = \sup_x (xy - zf(\omega, x))$.

$$xy \leq zf(\cdot, x) + \tilde{f}(\cdot, y, z), \ \forall x, y \in \mathbb{R}, \ \forall z \geq 0.$$

- $z > 0 \Rightarrow \tilde{f}(y, z) = zf^*(y/z)$.

Assumptions and Elementary Properties

- $\mathcal{P}$: set of prob's $\ll \mathbb{P}$ ($\Rightarrow \mathcal{P} \subset L^1(\mathbb{P})$).

$$I_{\mathcal{P},f}(X) := \sup_{P \in \mathcal{P}} E^P[f(\cdot, X)],\ X \in L^\infty$$

$$J_{\mathcal{P},\tilde{f}}(Y) := \inf_{P \in \mathcal{P}} E[\tilde{f}(\cdot, Y, dP/d\mathbb{P})],\ Y \in L^1.$$

(A1) $\mathcal{P}$ is convex and $\sigma(L^1, L^\infty)$-compact.

(A2) $\exists X \in L^\infty$ s.t. $\{f(\cdot, X)^+ dP/d\mathbb{P}\}_{P \in \mathcal{P}}$ is uniformly integrable.

(A3) $\forall P \in \mathcal{P}, \exists Y \in L^1$ s.t. $\tilde{f}(\cdot, Y, dP/d\mathbb{P})^+ \in L^1$.

- $I_{\mathcal{P},f}$, $J_{\mathcal{P},\tilde{f}}$ are well-defined, $I_{\mathcal{P},f}$ is LSC.

$$E[XY] \leq I_{\mathcal{P},f}(X) + J_{\mathcal{P},\tilde{f}}(Y),\ \forall X \in L^\infty,\ \forall Y \in L^1.$$

Robust Version of Rockafellar Theorem

- $\mathcal{D} := \{X \in L^\infty : \{f(\cdot, X)^+ dP/d\mathbb{P}\}_{P\in\mathcal{P}} \text{ is UI}\} \subset \mathrm{dom}(I_{\mathcal{P},f})$.

Main Abstract Theorem

For any $\nu \in ba$ with Yosida-Hewitt decomp. $\nu = \nu_r + \nu_s$,

$$J_{\mathcal{P},\tilde{f}}\left(\frac{d\nu_r}{d\mathbb{P}}\right) + \sup_{X\in\mathcal{D}} \nu_s(X) \le (I_{\mathcal{P},f})^*(\nu) \le J_{\mathcal{P},\tilde{f}}\left(\frac{d\nu_r}{d\mathbb{P}}\right) + \sup_{X\in\mathrm{dom}(I_{\mathcal{P},f})} \nu_s(X)$$

- If $\mathcal{P} = \{\mathbb{P}\}$, $\mathcal{D} = \mathrm{dom}(I_{\mathcal{P},f}) \Rightarrow$ equality.
- In general, the inclusion $\mathcal{D} \subset \mathrm{dom}(I_{\mathcal{P},f})$ can be strict.

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Ramifications: When "$=$"?

- OK if $\mathcal{D} = L^\infty$!! But when?
- f: deterministic $\Rightarrow f(X) \in L^\infty$ (recall f is $\mathbb{R}$-valued).
- Slightly more generally,

Elementary but Important Corollary

Suppose $\exists g \in C(\mathbb{R})$ and $W \in L^0$ s.t. $\{WdP/d\mathbb{P}\}_{P\in\mathcal{P}}$ is UI and

$$f(\omega, x) \leq g(x) + W(\omega).$$

Then $\mathcal{D} = L^\infty$, hence:

- $I_{\mathcal{P},f}$ is continuous on all of L^∞,
- $$\sup_{X\in L^\infty} (\nu(X) - I_{\mathcal{P},f}(X)) = \begin{cases} J_{\mathcal{P},\tilde{f}}(d\nu/d\mathbb{P}) & \text{if } \nu \text{ is } \sigma\text{-additive} \\ +\infty & \text{otherwise.} \end{cases}$$

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Back to the Robust Utility: Suitable Conditions on B

- Let $f(\cdot, x) := -U(-x + B) \Rightarrow u_{\mathcal{P},B}(X) = -I_{\mathcal{P},f}(-X)$

$$f(\omega, x) \leq -\frac{\varepsilon}{1+\varepsilon} U\left(-\frac{1+\varepsilon}{\varepsilon} x\right) - \frac{1}{1+\varepsilon} U(-(1+\varepsilon)B^-)$$

(B^-) $\exists \varepsilon > 0$, $\{U(-(1+\varepsilon)B^-) dP/d\mathbb{P}\}_{P\in\mathcal{P}}$ is UI.

$\Rightarrow \mathcal{D} = L^\infty$, i.e., $\{f(\cdot, X)^+ dP/d\mathbb{P}\}_{P\in\mathcal{P}}$ is UI, $\forall X \in L^\infty$.

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Back to the Robust Utility: Suitable Conditions on B

- Let $f(\cdot, x) := -U(-x + B) \Rightarrow u_{\mathcal{P},B}(X) = -I_{\mathcal{P},f}(-X)$

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Robust Utility Functional

(B^+) $\exists \varepsilon > 0,\ E^P[U(-\varepsilon B^+)] > -\infty,\ \forall P \in \mathcal{P}.$

- $\tilde{f}(\cdot, d\nu/d\mathbb{P}, dP/d\mathbb{P}) \in L^1 \Leftrightarrow V(\nu|P) < \infty$
- $V(P|P) = V(1) < \infty,\ \forall P \in \mathcal{P} \Rightarrow$ (A3).
- $\nu \in ba^\sigma_+,\ J_{\mathcal{P},\tilde{f}}(d\nu/d\mathbb{P}) = \begin{cases} V(\nu|\mathcal{P}) + \nu(B) & \text{if } V(\nu|\mathcal{P}) < \infty \\ +\infty & \text{otherwise.} \end{cases}$

Important Corollary implies:

Key Lemma.

Assume (B^-), (B^+), and $\mathcal{P}$ is compact.

1. $u_{\mathcal{P},B}$ is continuous on all of L^∞.
2. $\forall \nu \geq 0$,

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Duality for Robust Utility Maximization

Assumptions

- $U \in C^1(\mathbb{R})$, strictly concave & increasing with
  (Inada) $\lim_{x\to-\infty} U'(x) = +\infty$ & $\lim_{x\to+\infty} U'(x) = 0$
  (RAE) $\liminf_{x\searrow-\infty} \frac{xU'(x)}{U(x)} > 1$ & $\limsup_{x\nearrow\infty} \frac{xU'(x)}{U(x)} < 1.$
- S: d-dim., càdlàg $\mathbb{P}$-locally bounded semimartingale.
- $\mathcal{P}$: convex & weakly compact set of prob's $P \ll \mathbb{P}$.
- $\mathcal{M}_V^e \neq \emptyset$.

$$\mathcal{M}_V := \{Q \in \mathcal{M}_{\mathrm{loc}} : \ V(Q|\mathcal{P}) < \infty\}$$

- B satisfies (B^-) & (B^+).

Duality for Robust Utility Maximization with a Claim

Duality Theorem

$$\sup_{\theta\in\Theta_{bb}} \inf_{P\in\mathcal{P}} E^P[U(\theta\cdot S_T + B)] = \min_{\lambda>0, Q\in\mathcal{M}_V} \left(V(\lambda Q|\mathcal{P}) + \lambda E^Q[B]\right).$$

- Recall: $V(\nu|\mathcal{P}) = \inf_{P\in\mathcal{P}} V(\nu|P)$.
- The "min" is attained by $\exists(\hat{\lambda}, \hat{Q})$, but $\hat{Q} \not\ll \mathbb{P}$, in general.
- Duality is stable under change of Θ:

$$\Theta_V := \{\theta \in L(S) : \theta_0 = 0,\ \theta\cdot S \text{ is } Q\text{-superMG}, \forall Q \in \mathcal{M}_V\}$$

Duality remains true for $\Theta_{bb} \subset \forall\Theta \subset \Theta_V$.

- Robust version of utility indifference valuation.

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Application: Robust Utility Indifference Prices

Comparing the maximal robust utility

- $\sup_{\theta \in \Theta_{bb}} u_{\mathcal{P},B}(-p + \theta \cdot S_T)$: buy the claim B at the price p.
- $\sup_{\theta \in \Theta_{bb}} u_{\mathcal{P},0}(\theta \cdot S_T)$: not buy.
- Indifference Price $p(B)$: maximal acceptable price:

$$p(B) = \sup\{p : \sup_{\theta \in \Theta_{bb}} u_{\mathcal{P},B}(-p + \theta \cdot S_T) \geq \sup_{\theta \in \Theta_{bb}} u_{\mathcal{P},0}(\theta \cdot S_T)\}$$

Corollary

$$p(B) = \inf_{Q \in \mathcal{M}_V} (E^Q[B] + \gamma(Q)),$$

$$\gamma(Q) = \inf_{\lambda > 0} \left(V(\lambda Q|\mathcal{P}) - \inf_{\lambda' > 0, Q' \in \mathcal{M}_V} V(\lambda' Q'|\mathcal{P}) \right).$$

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Proof of Abstract Theorem

Outline

Second Inequality (Easy Part)

$$\sup_{X \in L^\infty} (\nu(X) - I_{\mathcal{P},f}(X)) \le J_{\mathcal{P},\tilde{f}}\left(\frac{d\nu_r}{d\mathbb{P}}\right) + \sup_{X \in \mathrm{dom}(I_{\mathcal{P},f})} \nu_s(X).$$

- Recall $E[XY] \le I_{\mathcal{P},f}(X) + J_{\mathcal{P},\tilde{f}}(Y)$.
- Note: $\nu(X) - I_{\mathcal{P},f}(X) = E[X(d\nu_r/d\mathbb{P})] - I_{\mathcal{P},f}(X) + \nu_s(X)$,

$$\begin{aligned}\sup_{X \in L^\infty} (\nu(X) - I_{\mathcal{P},f}(X)) &= \sup_{X \in \mathrm{dom}(I_{\mathcal{P},f})} (\nu(X) - I_{\mathcal{P},f}(X)) \\ &\le \sup_{X \in \mathrm{dom}(I_{\mathcal{P},f})} \left(J_{\mathcal{P},\tilde{f}}\left(\frac{d\nu_r}{d\mathbb{P}}\right) + \nu_s(X)\right) \\ &= J_{\mathcal{P},\tilde{f}}\left(\frac{d\nu_r}{d\mathbb{P}}\right) + \sup_{X \in \mathrm{dom}(I_{\mathcal{P},f})} \nu_s(X).\end{aligned}$$

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First Inequality 1: Lower Bound via Minimax

$$\sup_{X\in L^\infty}\left(\nu(X) - I_{\mathcal{P},f}(X)\right) \geq J_{\mathcal{P},\tilde{f}}(d\nu_r/d\mathbb{P}) + \sup_{X\in\mathcal{D}} \nu_s(X)$$

- $\mathcal{D} = \{X \in L^\infty : \{f(\cdot, X)^+ dP/d\mathbb{P}\}_{P\in\mathcal{P}} \text{ is UI}\} \neq \emptyset$ (by (B2).

$$\begin{aligned}
\sup_{X\in L^\infty}\left(\nu(X) - I_{\mathcal{P},f}(X)\right) &= \sup_{X\in L^\infty}\inf_{P\in\mathcal{P}}\left(\nu(X) - E^P[f(\cdot,X)]\right) \quad \text{(definition)}\\
&\geq \sup_{X\in\mathcal{D}}\inf_{P\in\mathcal{P}}\left(\nu(X) - E^P[f(\cdot,X)]\right) \overset{\text{minimax}}{=} \inf_{P\in\mathcal{P}}\sup_{X\in\mathcal{D}}\left(\nu(X) - E^P[f(\cdot,X)]\right)\\
&= \inf_{P\in\mathcal{P}}\sup_{X\in\mathcal{D}}\left\{\left(\nu_r(X) - E^P[f(\cdot,X)]\right) + \nu_s(X)\right\} \quad (\because \nu = \nu_r + \nu_s)\\
&\overset{??}{\geq} J_{\mathcal{P},\tilde{f}}(d\nu_r/d\mathbb{P}) + \sup_{X\in\mathcal{D}} \nu_s(X).
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- It suffices to show:

Claim.

$\forall \alpha < J_{\mathcal{P},\tilde{f}}(d\nu_r/d\mathbb{P})$ and $\forall \beta < \sup_{X \in \mathcal{D}} \nu_s(X)$,

$$\sup_{X \in \mathcal{D}} (\nu(X) - E^P[f(\cdot, X)]) > \alpha + \beta, \quad \forall P \in \mathcal{P}.$$

- Fix $X_s \in \mathcal{D}$ with $\nu_s(X) > \beta$.
- By singularity, $\exists (A_n) \subset \mathcal{F}$ s.t. $\mathbb{P}(A_n) \nearrow 1$ & $\nu_s(A_n) = 0$.
- Consequence: $\forall X \in L^\infty$, $\nu_s(1_{A_n} X + 1_{A_n^c} X_s) = \nu_s(X_s) > \beta$.
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Proof of Claim 1: Measurable Selection

- By $\alpha < \inf_{P \in \mathcal{P}} E[\tilde{f}(\cdot, d\nu_r/d\mathbb{P}, dP/d\mathbb{P})]$, $\exists Z_P \in L^1$ s.t.

$$E[Z_P] > \alpha,\ Z_P < \tilde{f}\left(\cdot, \frac{d\nu_r}{d\mathbb{P}}, \frac{dP}{d\mathbb{P}}\right) = \sup_{x \in \mathbb{R}} \left(x\frac{d\nu_r}{d\mathbb{P}} - \frac{dP}{d\mathbb{P}} f(\cdot, x) \right).$$

- A measurable selection theorem shows: $\exists X_P^0 \in L^0$ s.t.

$$Z_P \leq X_P^0 \frac{d\nu_r}{d\mathbb{P}} - \frac{dP}{d\mathbb{P}} f(\cdot, X_P^0)$$

- $\alpha < E[Z_P] \leq$ "$E[X_P^0 d\nu_r/d\mathbb{P}] - E^P[f(\cdot, X_P^0)]$".
- But $X_P^0 \notin \mathcal{D}$ ($\notin L^\infty$).

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Proof of Claim 1: Measurable Selection

- By $\alpha < \inf_{P \in \mathcal{P}} E[\tilde{f}(\cdot, d\nu_r/d\mathbb{P}, dP/d\mathbb{P})]$, $\exists Z_P \in L^1$ s.t.

$$E[Z_P] > \alpha,\ Z_P < \tilde{f}\left(\cdot, \frac{d\nu_r}{d\mathbb{P}}, \frac{dP}{d\mathbb{P}}\right) = \sup_{x \in \mathbb{R}} \left(x \frac{d\nu_r}{d\mathbb{P}} - \frac{dP}{d\mathbb{P}} f(\cdot, x)\right).$$

- A measurable selection theorem shows: $\exists X_P^0 \in L^0$ s.t.

$$Z_P \leq X_P^0 \frac{d\nu_r}{d\mathbb{P}} - \frac{dP}{d\mathbb{P}} f(\cdot, X_P^0)$$

- $\alpha < E[Z_P] \leq$ "$E[X_P^0 d\nu_r/d\mathbb{P}] - E^P[f(\cdot, X_P^0)]$".
- But $X_P^0 \notin \mathcal{D}$ ($\notin L^\infty$).

Proof of Claim 2: Final Step

- Recall: f is finite-valued, X_P^0 is $\mathbb{P}$-a.s. finite.
- $B_n := \{|X_P^0| \le n\} \cap \{|f(\cdot, X_P^0)| \le n\}$, and $C_n := A_n \cap B_n$.
- $\mathbb{P}(C_n) \nearrow 1$ & $\nu_s(C_n) = 0 \Rightarrow \nu_s(X_P^n) = \nu_s(X_s) > \beta$.
- $X_P^n := 1_{C_n} X_P^0 + 1_{C_n^c} X_s \in \mathcal{D}$

$$E[X_P^n d\nu_r/d\mathbb{P}] - E^P[f(\cdot, X_P^n)] \ge E[Z_P] + E[1_{C_n^c} \Xi_P]$$

- $\Xi_P = X_s d\nu_r/d\mathbb{P} - f(\cdot, X_s) dP/d\mathbb{P} - Z_P \in L^1 \Rightarrow E[1_{C_n^c} \Xi_P] \to 0.$
- $E[Z_P] > \alpha$, hence

$$\sup_{X \in \mathcal{D}} (\nu(X) - E^P[f(\cdot, X)] \ge E[Z_P] + \nu_s(X_s) > \alpha + \beta.$$

QED !!

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Thank You for Your Attention !!

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