ISOSTATIC GRAPHS AND SEMI-SIMPLICIAL MAPS

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1. Abstract

Following work initiated by Tiong Seng Tay, we know that a graph G is generically isostatic in dimension D if it has a shellable semi-simplicial map f to the D-simplex K_{d+1}. The converse is known only for $D = 1, 2$. We of course concentrate on the case $D = 3$.

In a semi-simplicial map to K_4, incidence must be preserved: edges whose end vertices go to the same vertex, say j, we call a *loop at* j, which must be sent to an edge of K_4 incident to j. We also insist that the inverse image $f^{-1}(ij)$ of any edge ij of K_4 be a *tree* connecting the vertex set $f^{-1}(i) \cup f^{-1}(j)$. A map is *shellable* if and only if, starting from the degenerate placement of the graph as its image under f, *in* the tetrahedron, the vertices can be gradually separated without making discontinuous change in edge directions.

A simple proof shows that the existence of a shellable semi-simplicial map $F : G \rightarrow K_4$ implies G is 3-isostatic. It is not known whether every 3-isostatic graph has such a shellable map. But "normally" there are so many you would not want to bother to count them.

Computer programs, which I write in Python, can establish that a given graph is isostatic, and quickly so, simply because so many shellable maps exist. But such programs are virtually useless in showing that a 3v-6 graph is dependent, since we must wait until the dreary end of the computation, only to find that there are no shellable maps, . . . and even then not to be sure of dependency, since there is no such theorem proven.

The situation is very similar to that encountered when using Henneberg reduction (and for good reason).

In this talk we describe recent work to cut down the search for maps, and to eliminate the need to produce shellings, by concentrating on maps in which non-shellable vertex packets (inverse images of a single vertex) can not occur: those in which vertex packets induce subgraphs having no cycles. Such maps are *"freely shellable"*.

Roger Poh has proven that for any planar graph there is a partition of its vertices into no more than three parts, each part inducing a path as subgraph. Such partitions are not necessarily vertex partitions under inverse image of a semi-simplicial map. But since maximal planar graphs are generically 3-isostatic, we are still led to conjecture that isostatic graphs share similar properties. We conjecture:

A graph G is generically 3-isostatic graph if and only if it has a semi-simplicial map to the tetrahedron in which all vertex packets induce subgraphs that are broken paths (forests with all vertices of valence 1 or 2).

This conjecture is false if you insist that the induced subgraphs be paths. As an example, consider a hinged ring of six tetrahedra.

Time permitting (or on another occasion during the week) we should discuss computational methods for constructing the generically 3-rigid *components* of an arbitrary graph, or for finding dependencies.

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