Characterizing graphs with Gram dimension at most four

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Given a graph $G = (V = [n], E)$, its *Gram dimension* $\text{gram}(G)$ is the smallest integer $r \geq 1$ such that, for any $n \times n$ positive semidefinite matrix X, there exist vectors $p_i \in \mathbb{R}^r$ $(i \in V)$ satisfying $X_{ij} = p_i^T p_j$ for all $ij \in V \cup E$.

The class of graphs with Gram dimension at most r is closed under taking minors and clique sums. Clearly, K_{r+1} is a minimal forbidden minor for membership in this class. We show that this is the only minimal forbidden minor for $r \leq 3$ while, for $r = 4$, there are two minimal forbidden minors: the complete graph K_5 and the octahedron $K_{2,2,2}$.

These results are closely related to the characterization of Belk and Connelly (2007) for the class of d-realizable graphs with $d \leq 3$. Recall that G is *d-realizable* if, for any vectors u_i $(i \in V)$, there exist other vectors $v_i \in \mathbb{R}^d$ $(i \in V)$ satisfying $\|u_i - u_j\|_2 = \|v_i - v_j\|_2$ for all $ij \in E$; that is, for any $n \times n$ Euclidean distance matrix, the distances corresponding to edges can be realized in $\mathbb{R}^d$. Denoting by $\text{edm}(G)$ the smallest integer d such that G is d-realizable, the two parameters are related by $\text{gram}(G) = \text{edm}(\nabla G)$, where ∇G is the one-node suspension of G. Moreover, they satisfy: $\text{gram}(\nabla G) = \text{gram}(G) + 1$ and $\text{edm}(\nabla G) \geq \text{edm}(G) + 1$. Hence, $\text{gram}(G) \geq \text{edm}(G) + 1$, so that our results imply those of Belk and Connelly.

Joint work with Antonis Varvitsiotis (CWI, Amsterdam).