The Work of Mike Shub in Complexity

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(P) To analyze (in terms of cost) the behavior of specific algorithms (meant to solve specific problems).

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(3) Conditioning of Numerical Problems.

Zeros of Polynomial Systems

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on the average: w.r.t. a Gaussian distribution on the input f.

$$D := \max\{d_1, \ldots, d_n\} \qquad\qquad N \approx n\binom{D+n}{n}$$

Adaptive linear homotopy

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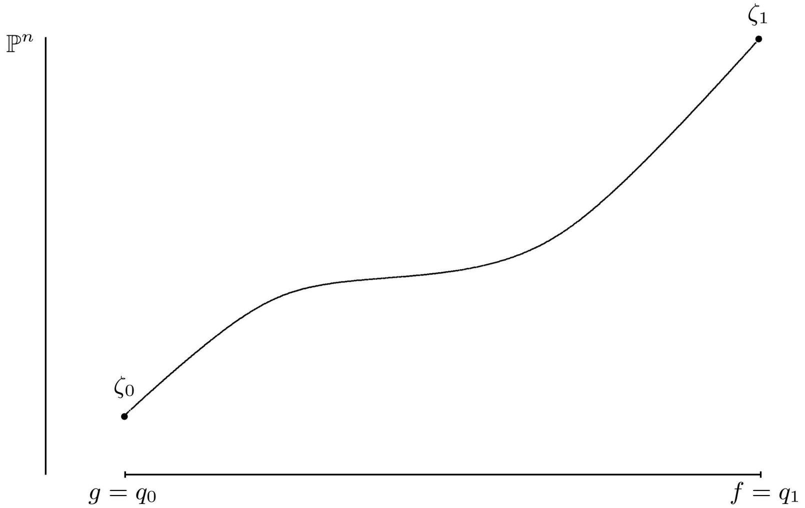
$$q_t := (1-t)g + tf \qquad \text{for } t \in [0,1].$$

- If no q_t has a multiple zero, then there exists a unique lifting of this segment to a curve

$$t \in [0,1] \mapsto (q_t, \zeta_t)$$

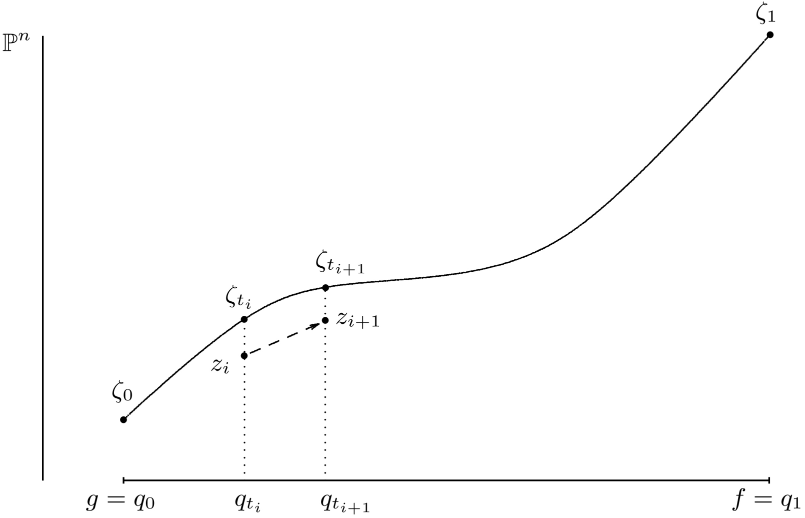
such that $\zeta_0 = \zeta$. Since $q_1 = f$, ζ_1 is a zero of f.



$\mathbb{P}^n$
ζ_1
ζ_0
$g = q_0$
$f = q_1$

The idea is to follow this curve numerically: partition $[0, 1]$ into $t_0 = 0, \ldots, t_k = 1$. Writing $q_i := q_{t_i}$, successively compute approximations z_i of ζ_{t_i} by Newton's method starting with $z_0 := \zeta$. More specifically, compute

$$z_{i+1} := N_{q_{i+1}}(z_i).$$



$\mathbb{P}^n$
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ζ_{t_i}
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(2) How large should $d(q_{i+1}, q_i)$ be?

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- We compute t_{i+1} adaptively from t_i such that

$$d(q_{i+1}, q_i) = \frac{0.0085}{D^{3/2}\, \mu_{\mathrm{norm}}^2(q_i, z_i)}.$$

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“Bézout VI” (**M.S.**, *Found. Comput. Math. 2009*)

For all i, z_i is an approximate zero of q_i. In particular z_K is an approximate zero of f. Moreover,

$$K(f, g, \zeta) \leq 217\, D^{3/2}\, d(f, g) \int_0^1 \mu_{\mathrm{norm}}^2(q_\tau, \zeta_\tau)\, d\tau.$$

Here $\tau \in [0, 1]$ is a ratio of angles and not of Euclidean distances.

This result relates to cost in a clear manner. Each Newton step takes $\mathcal{O}(N)$ arithemetic operations. Therefore, the total number of such operations performed along the homotopy is $\mathcal{O}(N\,K(f,g,\zeta))$.

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(2) a deterministic algorithm working in near-polynomial time (average polynomial time for all but a few pairs (n, D) and average time $N^{\mathcal{O}(\log\log N)}$ on those pairs). [P. Bürgisser – F.C.].

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• Back to the roots? [D. Armentano, **M.S.**]

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  QUAD Given $f_1, \ldots, f_m$ in $\mathbb{C}[X_1, \ldots, X_n]$ of degree 2, is there a $\xi \in \mathbb{C}^n$ such that $f_1(\xi) = \ldots = f_m(\xi) = 0$?

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average-case vs worst-case

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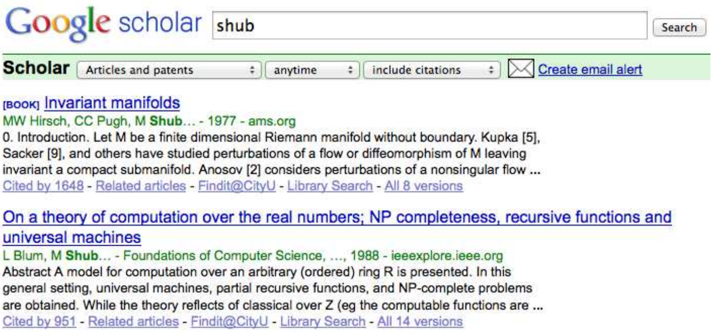

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For every $d \geq 1$

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Conditioning of Numerical Problems

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- Condition numbers have also been used in estimates for the speed of convergence of iterative algorithms (complexity!).

Mike's first work in conditioning studies a notion of condition number obtained by replacing "worst-case perturbation" by "average perturbation." This is relevant for finite-precision analyses.

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Then attention turned to the relationship between condition and complexity. This relationship pervades the Bézout series.

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The main result in Bézout VI allows one to use instead

$$\mu_{\text{av}}(f) := \sqrt{\frac{1}{\mathcal{D}} \sum_{i \le \mathcal{D}} \mu^2_{\text{norm}}(f, \zeta_i)}.$$

This fact is, as we already pointed out, at the core of the recent advances towards a final solution to Smale's 17th problem.

A Unifying Theory?

c) For a decision problem $\mathbb{R}^\infty, \mathbb{R}^\infty_{yes}$ we say a problem instance $x \in \mathbb{R}^k$ is ill posed if $\forall \delta > 0$ $\exists\, y_1, y_2 \in \mathbb{R}^k$ with $\|x - y_i\| \le \delta\|x\| + \delta$ $\quad i = 1, 2$ and $y_1 \in \mathbb{R}^k_{yes}$ while $y_2 \in \mathbb{R}^k_{no}$. We denote the set of ill-posed problems by Ill and Ill_k for inputs of size k if confusion is likely.