Solving a Real Analogue of Smale's $17^{\underline{\text{th}}}$ Problem

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Applications of Solving Real Polynomial Systems

- Maximum Likelihood Estimation in UQ

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- Satellite Orbit Design, Geometric Modelling...
- Refined bounds help in complexity theory

Motivating Problem/Theorem

Consider $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ with maximal degree D.

Smale's 17th Problem

Can a zero of n complex polynomial equations

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Can a zero of n complex polynomial equations in n unknowns be found approximately,

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Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?

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Can you go faster over the reals?

Warm-Up: One Variable, Three Terms

Can you decide whether

$$1 + cx_1^d + x_1^D \qquad (0 < d < D)$$

has 0, 1, or 2 positive roots, using a number of bit operations **sub-linear** in $D + \log c$?

Warm-Up: One Variable, Three Terms

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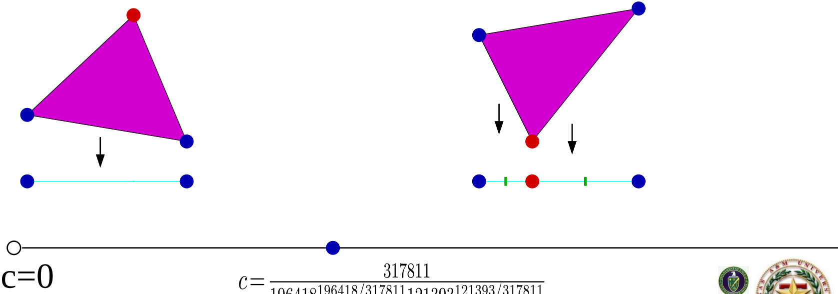
which can be done in polynomial time via **Baker's Theorem on Linear Forms in Logarithms [1967]**!
i.e., you can attain complexity $\log^{O(1)}(Dc)$
[Bihan, Rojas, Stella, 2009].

Discriminant Chambers and Liftings

We call any connected component of the **complement** of

$$\{c \in \mathbb{R} \setminus \{0\} \mid \bar{\Delta}_{\{0,d,D\}}(c) = 0\}$$

a **(reduced) discriminant chamber**.



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$$\approx 1.944526275\ldots$$

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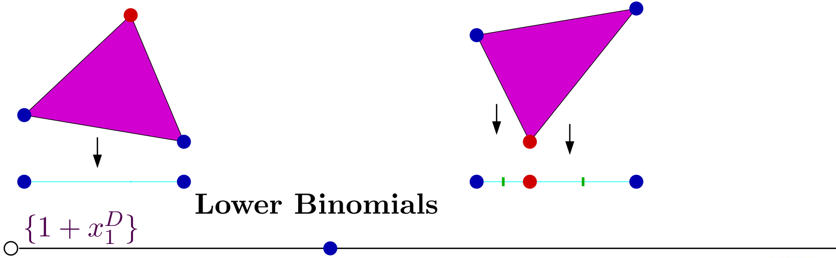

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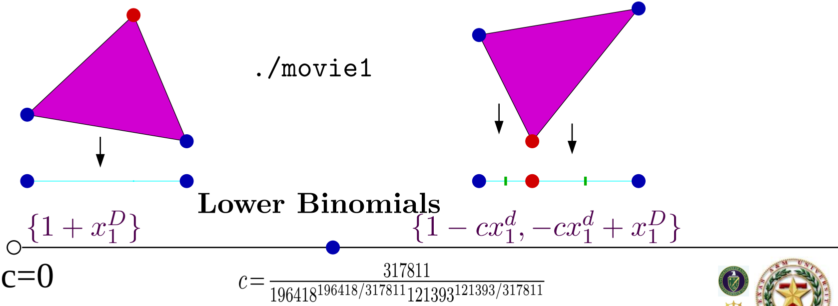

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One Variable Not So Trivial

Counting the roots of $f(x_1) := 1 + ax_1^{14} + bx_1^{2^{129}} + cx_1^{2^{2013}} + x^D$ can be done within $(D + \log(abc))^{O(1)}$ bit operations [Sturm, 1835],

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Indeed, while f has no more than 8 **real** roots, computational algebra and numerical algebraic geometry do not (yet) give us such a speed-up. However, you can go faster if you use $\mathcal{A}$-discriminants...

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Faster Real Root Counting (One Variable)

Theorem (Ascher, Avendano, Rojas, Rusek, 2012)

For any finite subset $\mathcal{A} \subset \mathbb{Z}$ *of cardinality* $1+k$ *and maximum coordinate absolute value* D*, there is a subset* $\mathcal{S}_{\mathcal{A}} \subseteq \mathbb{R}^{1+k}$

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Chambers Can Be Complicated...

./movie2

Sparsity in General

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For instance, the *supports of*

$$x_1^{2012} x_3^{-1} + \frac{44}{31} x_2^{1006} x_3^{-1} - 1$$
$$x_2^{2012} x_1^{-1} - \sqrt{12} x_3^{1006} x_1^{-1} - 1$$
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all lie in

$\{(0,0,0),(2012,0,-1),(0,1006,-1),(-1,2012,0),(-1,0,1006),(0,-1,2012),(1006,-1,0)\}$

and we thus have a 7-nomial 3×3 system.

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This system has exactly $8,144,865,727$ complex roots but no more than 124 roots in $\mathbb{R}^3_+$.

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Sparse Size

Consider $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$, each having exponent vectors contained in the same set of $n+k$ points. We call F a *(real)* $(n+k)$*-nomial* $n \times n$ *system*.

For instance, the *supports of*

$$F := \begin{cases} x_1^{2D} x_3^{-1} + a x_2^D x_3^{-1} \pm 1 \\ x_2^{2D} x_1^{-1} + b x_3^D x_1^{-1} \pm 1 \\ x_3^{2D} x_2^{-1} + c x_1^D x_2^{-1} \pm 1 \end{cases}$$

all lie in $\{(0,0,0), (D,-1,0), \ldots\}$ and we thus have a 7-nomial 3×3 system.

This system has exactly $8D^3 - 1$ complex roots but no more than 124 roots in $\mathbb{R}^3_+$.

size(F) here is $O(\log(D) + \log(a) + \log(b) + \log(c))$.

Motivating Conjecture 1

Consider $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$, each having exponent vectors contained in the same set of $n + k$ points. Let $\Omega(n, k)$ denote the maximal number of nondegenerate roots in $\mathbb{R}^n_+$ over all such F.

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Local Fewnomial Conjecture (real case)

There are absolute constants $C_1, C_2 > 0$ such that for all $n, k \geq 1$, we have $(n+k-1)^{C_1 \min\{n,k-1\}} \leq \Omega(n,k) \leq (n+k-1)^{C_2 \min\{n,k-1\}}$.

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True for $n=1$ [Descartes, 1637] and $k=1$ [anon]. Evidence in general comes from [Khovanski, 1980s], [Rojas, 2004], [Bihan & Sottile, 2007], [Bihan, Rojas, Sottile, 2007], and [Avendaño, Pébay, Rojas, Rusek, & Thompson, 2012].

Real Analogue I of Smale's 17th Problem

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Conjecture (Exact Counting over $\mathbb{R}$)

Suppose we consider random F *with maximum exponent coordinate* D. *Then there is a uniform algorithm that, in time polynomial in* $\Omega(n,k) + \log D$, *computes a positive integer that, with high probability, is exactly the number of roots of* F *with all coordinates positive.*

2×2 Trinomial Systems

The discriminant polynomial $\Delta(a,b)$ for

$$\widetilde{F} := \begin{cases} y^{41} + ax^{82} - x^{82}y^{40} \\ x^{41} + by^{82} - x^{40}y^{82} \end{cases}$$

has degree 23206 and coefficients having thousands of digits: hopeless on any computer algebra system...

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...however, the **Horn-Kapranov Uniformization** parametrizes the underlying zero set with a **one-line formula!**

Horn-Kapranov Uniformization

Succinctly,

$$F := \begin{cases} a_1 y^{41} + a_2 x^{82} + a_3 x^{82} y^{40} \\ b_1 x^{41} + b_2 y^{82} + b_3 x^{40} y^{82} \end{cases},$$

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where $\mathcal{A} = \begin{bmatrix} 0 & 82 & 82 & 41 & 0 & 40 \\ 41 & 0 & 40 & 0 & 82 & 82 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ and the columns of B are any basis for the right nullspace of $\widehat{\mathcal{A}} := \begin{bmatrix} 1 & \cdots & 1 \\ & \mathcal{A} & \end{bmatrix}$.

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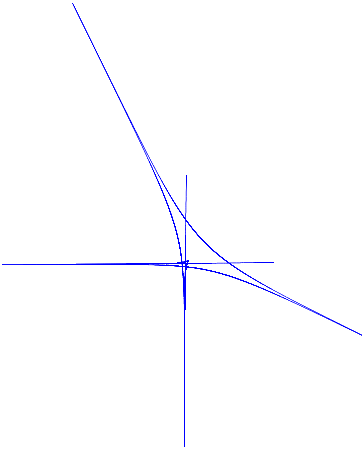
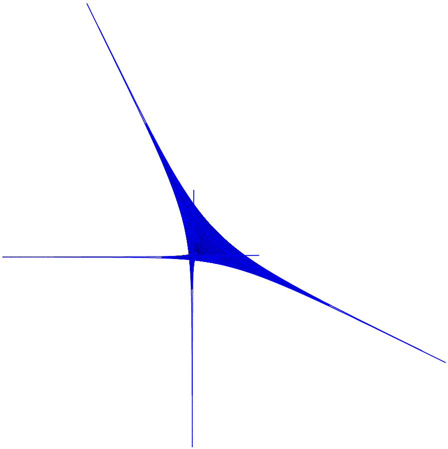
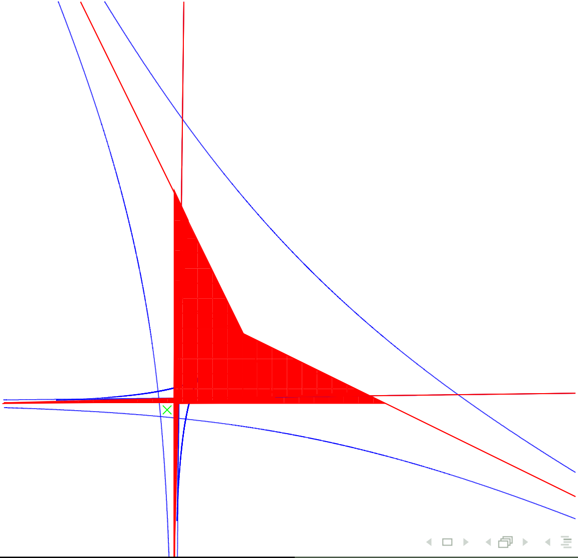
So let's consider the *amoeba* of $\nabla_{\mathcal{A}}(\mathbb{R})$, i.e., the image of the real part under $\mathrm{Log}|\cdot|$...

Inner/Outer Chambers





Walls and Chamber Cuttings



Cutting Complex in Higher Dimensions

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Lower Binomial Systems

./movie4

One Sparse Multivariate Polynomial

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For any finite subset $\mathcal{A} \subset \mathbb{Z}^n$ of cardinality $n+k$ and maximum coordinate absolute value D, there is a subset $\mathcal{S}_{\mathcal{A}} \subseteq \mathcal{F}_{\mathcal{A}}$ with stable log-uniform content 1 such that, on any input $F \in \mathcal{S}_{\mathcal{A}}$, one can count exactly the number of connected components of the positive zero set of F

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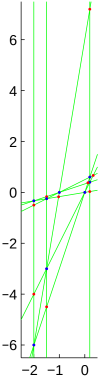
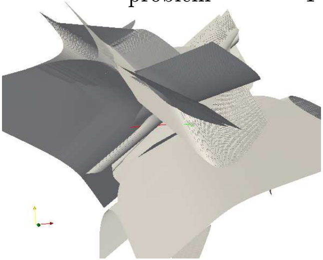
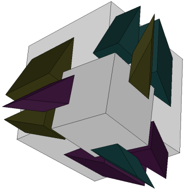
Idea

Polynomial System $\longrightarrow$ Exponent Vectors yield data structure $\longrightarrow$ Coefficients yield point location problem $\longrightarrow$ Optimal Start Systems for Polyhedral Homotopy

$$x^{2D} + ay^{D} - z$$
$$y^{2D} + bz^{D} - x$$
$$z^{2D} + cx^{D} - y$$







$\text{Log}|\nabla_{\mathcal{A}}|$ as a surface

...let's see how the complement of the last line arrangement parametrizes the underlying $\mathcal{A}$-discriminant amoeba...

`./movie5`

(thanks to Korben Rusek)

♡ Thank you for listening!

...and Happy Birthday Mike!

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- **abc implies:** (1) Effective Falting's Theorem [Elkies, 1991], (2) Effective* Roth's Theorem [Bombieri, 1994; Surroca 2007], (3)* non-existence of Siegel zeroes for certain L-functions [Granville, 2000]. Conversely, suitably sharp versions of (1) or (2) imply variants of abc!