Damping-induced self-recovery phenomenon in mechanical systems with an unactuated cycle variable

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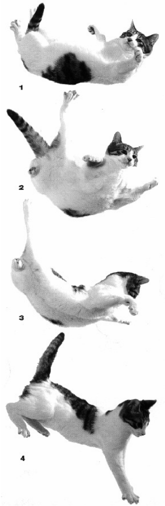
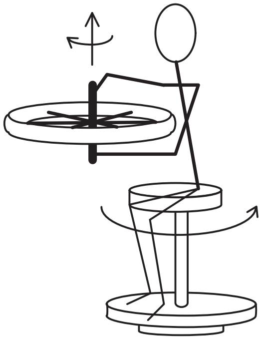
Angular Momentum Conservation

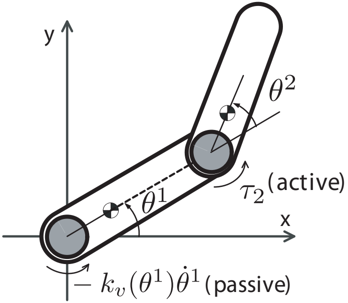


Thought Experiment



$$I_s\omega_s + I_w\omega_w = 0.$$

Horizontally Planar 2-Link Arm



$$I_i\omega_i + I_o\omega_o = 0.$$

Horizontally Planar 2-Link Arm: With or Without Damping

without damping

with damping

Horizontally Planar 2-Link Arm: Global Self-Recovery

Self-recovery is global, remembering the winding number.

Mechanical System with an Unactuated Cyclic Variable

- Configuration space Q = open subset of $\mathbb{R}^n$.
- Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j - V(\mathbf{q})$ with cyclic variable q^1

$$\frac{\partial L}{\partial q^1} = 0.$$

- Equations of Motion (EL equations with forces):

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^1} \qquad\qquad &= -k_v(q^1)\dot{q}^1 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} &= u_a, \quad a = 2, \ldots, n \end{aligned}$$

where

  - $-k_v(q^1)\dot{q}^1$ is a viscous damping force
  - $u_2, \ldots, u_n$ are control forces

Mechanical System with an Unactuated Cyclic Variable

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- Without damping ($k_v = 0$)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^1} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}^1} = \text{conserved}.$$

Mechanical System with an Unactuated Cyclic Variable

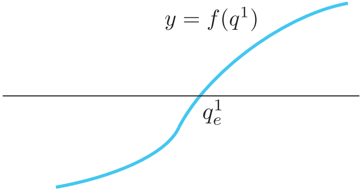
- Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j - V(\mathbf{q})$ with cyclic variable q^1 such that
$$\frac{\partial L}{\partial q^1} = 0.$$
- Equations of Motion (EL equations with forces):
$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^1} &= -k_v(q^1) \dot{q}^1 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} &= u_a, \quad a = 2, \ldots, n \end{aligned}$$
- New conserved quantity with damping
$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^1} + \int_0^{q^1} k_v(x) dx \right) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^1} + k_v(q^1) \dot{q}^1 = 0$$
$$\Rightarrow \underbrace{\frac{\partial L}{\partial \dot{q}^1} + \int_0^{q^1} k_v(x) dx}_{\text{damping-added momentum}} = \text{conserved}$$
- $\int_0^{q^1(t)} k_v(x) dx = \int_0^t k_v(x) \dot{x} dt = (-)$ impulse due to friction.

Damping-Induced Self-Recovery Phenomenon

Theorem (Chang and Jeon [2013, ASME J. DSMC]) Let

$$\mu = \frac{\partial L}{\partial \dot{q}^1} + \int_0^{q^1} k_v(x)dx = m_{1i}(\mathbf{q}(t))\dot{q}^i(t) + \int_0^{q^1(t)} k_v(x)dx.$$

Let $f(q^1) = \int_0^{q^1} k_v(x)dx - \mu$ such that



Suppose controls $u_a(t)$'s $(a = 2, \ldots, n)$ are chosen such that $q^a(t)$'s $(a = 2, \ldots, n)$ are bounded and $\lim_{t\to\infty} \dot{q}^a(t) = 0$ for all $a = 2, \ldots, n$. Then,

1. $lim_{t\to\infty} q^1(t) = q_e^1$.
2. If the initial condition is such that $\dot{q}^i(0) = 0$ for all $i = 1, \ldots, n$, then $lim_{t\to\infty} q^1(t) = q^1(0)$.

Sketch of Proof for Constant k_v with $\mu = 0$

Equation for q^1

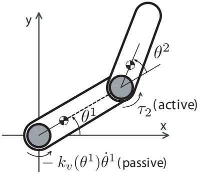
$$
\begin{aligned}
0 &= m_{1i}\dot{q}^i(t) + \int_0^{q^1} k_v dx = m_{1i}\dot{q}^i(t) + k_v q^1 \\
\Rightarrow \dot{q}^1 &= -\frac{k_v}{m_{11}} q^1 + \left(-\frac{1}{m_{11}} \sum_{a=2}^{n} m_{1a}\dot{q}^a \right),
\end{aligned}
$$

where $\dot{q}^i(0) = 0$ for all $i = 1, \ldots, n$ and $q^1(0) = 0$. Hence,

$$
\lim_{t\to\infty} \dot{q}^a = 0 \quad \forall a = 2, \ldots, n \Rightarrow \lim_{t\to\infty} q^1(t) = 0 = q^1(0).
$$

Remark: Damping coefficient $k_v(q^1)$ does not have to be a non-negative function. For example, $k_v(q^1) = 1 + 4\cos(q^1)$ shows self-recovery for $\mu = 0$.

Damping-Induced Bound



Suppose

$$\lim_{q^1 \to \infty} \int_0^{q^1} k_v(x)dx = \infty, \qquad \lim_{q^1 \to -\infty} \int_0^{q^1} k_v(x)dx = -\infty.$$

If controls $u_a(t)$'s are chosen such that $m_{11}(\mathbf{q}(t))$ is bounded above and below by two positive numbers and $m_{1a}(\mathbf{q}(t))$'s and $\dot{q}^a(t)$'s are bounded where $a = 2, \ldots, n$, then $q^1(t)$ is also bounded.

Damping-Induced Bound for Horizontally Planar 2-Link Arm

The motion of Link 1 (θ^1) is bounded when $\dot{\theta}^2$ is bounded.

Real Experiment

Several Unactuated Cyclic Variables

Link 2 (θ^2) is actuated and Links 1 and 3 (θ^1, θ^3) are unactuated but under friction.

Self-Recovery Seems to Occur Only for Linear Friction

Cubic friction $F = -kv^3$.

Summary

- Viscous damping force breaks symmetry, so the corresponding momentum is no longer conserved.
- Exists a new conserved quantity called *damping-added momentum*.
- Damping-induced self-recovery is global.
- Damping puts a bound on range of the unactuated variable.
- References:

  - D.E. Chang and S. Jeon, "Damping-induced self recovery phenomenon in mechanical systems with **an** unactuated cyclic variable," ASME Journal of Dynamic Systems, Measurement, and Control, 135(2), 2013. http://dx.doi.org/10.1115/1.4007556
  - D.E. Chang and S. Jeon, "On the damping-induced self-recovery phenomenon in mechanical systems with **several** unactuated cyclic variables," J. Nonlinear Science, Submitted. http://arxiv.org/abs/1302.2109