Directional Localization and Toral Eigenfunctions

Melissa Tacy

Department of Mathematics
Northwestern University

mtacy@math.northwestern.edu

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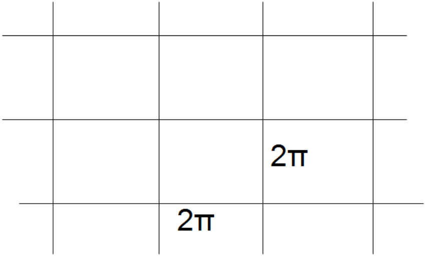
Laplacian Eigenfunctions

Let u be an eigenfunctions on smooth, compact, boundryless Riemannian manifold (M, g)

$$\Delta u = \lambda^2 u$$

What are the L^p growth properties of u

Suppose $M = \mathbb{T}^2$ how can be include algebraic information into analytic estimates?



Quantum mechanics

Eigenfunctions arise as stationary state of the Schrödinger equation

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - \Delta\right)\psi(t,x) = 0$$

$$\psi(t,x) = e^{itE}u$$

$$\Delta u = E$$

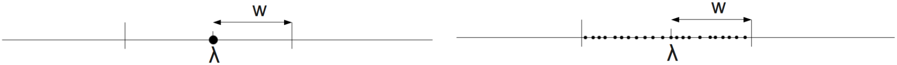
$\lambda^2 = E$ physically is interpreted as energy of the system. Want to study the high energy limit $\lambda \to \infty$

Spectral Clusters

Due to the uncertainty principle it is difficult to study one eigenfunction directly. We study norms of spectral clusters on windows of width w

$$E_\lambda = \sum_{\lambda_j \in [\lambda - w, \lambda + w]} E_j$$

E_j projection onto λ_j eigenspace.



Obviously include eigenfunctions but also can include sums of eigenfunctions if w is large enough. The smaller the window size the closer cluster estimates become to true eigenfunction estimates.

Quasimodes

In the semiclassical setting we study approximate eigenfunctions or quasimodes

$$(h^2\Delta - 1)u = hwf$$

where $\|f\|_{L^2} = O(1)$, same as studying width w windows

$$(\Delta - \lambda^2) \sum_{\lambda_j \in [\lambda - w, \lambda + w]} c_j u_j = \sum_{\lambda_j \in [\lambda - w, \lambda + w]} c_j (\lambda + \lambda_j)(\lambda - \lambda_j) u_j$$

Divide by $\lambda^2 = h^{-2}$

$$\sum_{\lambda_j \in [\lambda - w, \lambda + w]} c_j \frac{(\lambda + \lambda_j)(\lambda - \lambda_j)}{\lambda^2} u_j = O_{L^2}(\lambda^{-1} w) = O_{L^2}(hw)$$

So

Width w clusters $\rightarrow$ Quasimodes of order hw

Laplacian Eigenfunctions as Stationary States

We work in semiclassical setting with $h = \lambda^{-1}$

$$(hD_t - h^2\Delta)\psi(t,x) = 0$$

$$\psi(t,x) = e^{\frac{it}{h}} u(x)$$

$$(h^2\Delta - 1)u = 0$$

Use this formulation to express eigenfunction as a time average.
Quasimodes of order hw

$$(h^2\Delta - 1)u = hwf(x)$$

$$(hD_t - h^2\Delta)u = hwe^{\frac{it}{h}} f(x)$$

Invariance under propagation

Use the propagator $U(t) = e^{ith\Delta}$

$$\begin{cases} (hD_t - h^2\Delta)U(t) = 0 \\ U(0) = \mathrm{Id} \end{cases}$$

We write

$$\psi(t,x) = e^{\frac{it}{h}}u(x) = e^{ith\Delta}u(x) + \frac{1}{h}\int_0^t e^{i(t-s)h\Delta}[hwe^{\frac{is}{h}}f(x)]ds$$

$$u(x) = e^{-\frac{it}{h}}e^{ith\Delta}u(x) + we^{-\frac{it}{h}}\int_0^t e^{i(t-s)h\Delta}[e^{\frac{is}{h}}f(x)]ds$$

Can average this over times up to order $1/w$.

Propagation times

- Averaging over short times has the benefit of keeping the analysis local however we are then unable to tell the difference between good and bad quasimodes
- Longer time averages will differentiate between quasimodes however loss of locality

  For $w = 1$ Sogge

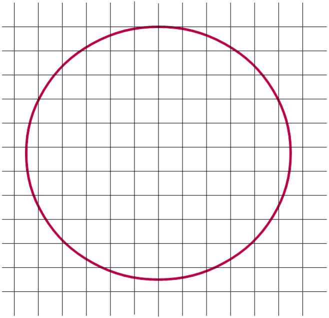
$$\|u\|_{L^p} \lesssim \lambda^{\delta(n,p)} \|u\|_{L^2}$$

$$\delta(n,p) = \begin{cases} \frac{n-1}{2} - \frac{n}{p} & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{4} - \frac{n-1}{2p} & 2 \leq p \leq \frac{2(n+1)}{n-1} \end{cases}$$

  Sharp on the sphere

Toral Eigenfunctions

Special case when $M = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$.
Eigenfunctions are the plane waves $e^{i\lambda k \cdot x}$. Periodicity requires that λk_1 and λk_2 are integers. So multiplicity is equal to the number intersections of the circle of radius λ and the integer lattice.



This is known to be $C_\epsilon \lambda^\epsilon$. So trivially we have better estimates for $\mathbb{T}^2$. Are L^p norms ever bounded? Zygmund

$$\|u\|_{L^4} \leq 5^{1/4} \|u\|_{L^2}$$

$e^{ith\Delta}$ on the Torus

Will develop $\widetilde{U}(t) = e^{ith\Delta_{\mathbb{R}^2}}$ in the form

$$\widetilde{U}(t)u = \int \tilde{e}(t,x,y)u(y)dy$$

Then let Γ be the set of translations

$$U(t)u = \sum_{\gamma\in\Gamma} \int \tilde{e}(t,x,\gamma y)u(y)dy$$

Will find that this sum is finite as $\tilde{e}(t,x,\gamma y)$ is supported when $d(x,\gamma y) \leq 1/w$.

The propagator $e^{ith\Delta_{\mathbb{R}^2}}$

We want to solve the evolution equation

$$\begin{cases} (hD_t - \Delta)\widetilde{U}(t) = 0 \\ \widetilde{U}(0) = \mathrm{Id} \end{cases}$$

Seek a solution of the form

$$\widetilde{U}(t)u = \int \tilde{e}(t,x,y)u(y)dy$$

$$\tilde{e}(t,x,y) = h^{-2} \int e^{\frac{i}{h}\phi(t,x,y,\xi)} a(x,\xi)d\xi$$

This is easy to solve can check that

$$\tilde{e}(t,x,y) = h^{-2}\int e^{\frac{i}{h}(<x-y,\xi>+t\xi\cdot\xi)}d\xi$$

is a solution

$$hD_t\tilde{e}(t,x,y) = h^{-2}|\xi|^2\int e^{\frac{i}{h}(<x-y,\xi>+t\xi\cdot\xi)}d\xi$$

$$hD_{x_i}\tilde{e}(t,\xi,y) = h^{-2}\xi_i\int e^{\frac{i}{h}(<x-y,\xi>+t\xi\cdot\xi)}d\xi$$

so

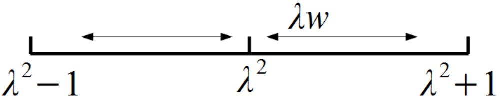
$$(hD_t - h^2\Delta)\tilde{e}(t,x,y) = 0$$

and

$$\tilde{e}(0,x,y) = h^{-2}\int e^{\frac{i}{h}<x-y,\xi>}d\xi$$

Choosing quasimode order

We want to choose a w so that we get no pollution from eigenfunctions with similar eigenvalue
Toral eigenfunctions $e^{i\lambda k \cdot x}$ where λk is a integer lattice point.
Therefore $\lambda^2 \in \mathbb{Z}$.



$$(\lambda \pm w)^2 = \lambda^2 \pm \lambda w + w^2$$

Need to choose $w = \lambda^{-1}$ or in semiclassical notation $w = h$

Quasimodes on the torus

We will assume we are working with an order h^2 quasimode (equivalent to $w = h$). We can propagate for times up to h^{-1}

$$u(x) = h\int \chi(ht)e^{-\frac{it}{h}}e^{ith\Delta}u(x)dt + h^2\int \chi(ht)e^{-\frac{it}{h}}\int_0^t e^{i(t-s)h\Delta}[e^{\frac{is}{h}}f(x)]dsdt$$

where $\chi(t)$ is supported in $\epsilon \leq t \leq 2\epsilon$. Focus on first term

$$h\int \chi(ht)e^{-\frac{it}{h}}e^{ith\Delta}udt = h\sum_{\gamma\in\Gamma}\int e^{-\frac{it}{h}}\tilde{e}(t,x,\gamma y)\chi(ht)u(y)dtd\xi dy$$

Will use stationary phase to simplify

$$\int e^{-\frac{it}{h}}\tilde{e}(t,x,\gamma y)\chi(ht)u(y)dtd\xi dy$$

for each γ

$$\int e^{-\frac{it}{h}}\tilde{e}(t,x,\gamma y)\chi(ht)dtd\xi = h^{-2}\int e^{\frac{i}{h}(<x-\gamma y,\xi>+t\xi\cdot\xi-t)}\chi(ht)dtd\xi$$

Stationary phase in (t,ξ)

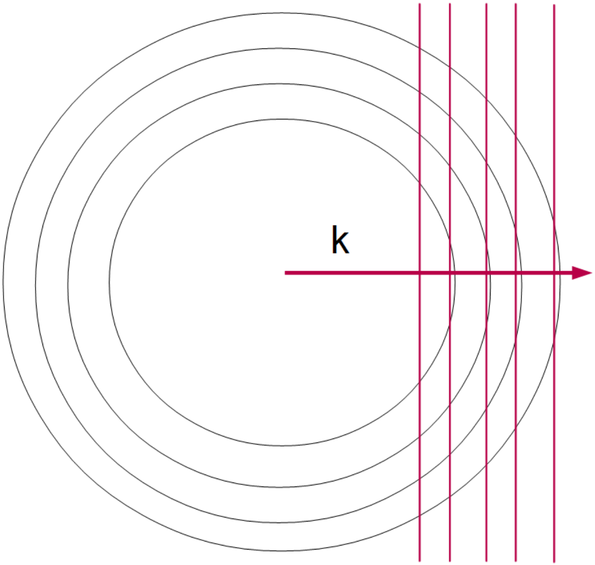
$$\xi\cdot\xi = 1$$

$$x-\gamma y = 2t\xi$$

$$\int e^{-\frac{it}{h}}\tilde{e}(t,x,\gamma y)\chi(ht)u(y)dtd\xi dy = h^{-1/2}\int e^{\frac{i}{h}|x-\gamma y|}a(x,y)u(y)dy$$

where $a(x,y)$ is supported $\epsilon h^{-1}\le|x-y|\le 2\epsilon h^{-1}$

$$u(x) = h^{1/2}\sum_{\gamma\in\Gamma}\int e^{\frac{i}{h}|x-\gamma y|}a(x,\gamma y)u(y)dy$$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Directional localization

We write

$$u(x) = Tu$$

$$Tu = h^{1/2} \sum_{\gamma \in \Gamma} \int e^{\frac{i}{h}|x-\gamma y|} a(x, \gamma y) u(y) dy$$

Definition

Let $\xi \in S^1$ and $\zeta : \mathbb{R}^2 \to \mathbb{R}^+$ a smooth cut off function supported in $|\eta| \leq 2$. Let T_ξ be given by

$$T_\xi u = h^{1/2} \sum_{\gamma \in \Gamma} \int e^{\frac{i}{h}|x-\gamma y|} a(x, \gamma y) \zeta \left(\frac{1}{h} \left(\frac{x - \gamma y}{|x - \gamma y|} - \xi \right) \right) u(y) dy$$

We say T_ξ is the component of T localized in direction ξ

Algebraic to analytic

Consider

$$T_\xi e^{\frac{i}{h}k\cdot x}$$

where $|\xi - k| \geq h^{1-\epsilon}$

$$h^{1/2}\sum_{\gamma\in\Gamma}\int e^{\frac{i}{h}|x-\gamma y|+k\cdot y}a(x,\gamma y)\zeta\left(\frac{1}{h}\left(\frac{x-\gamma y}{|x-\gamma y|}-\xi\right)\right)dy$$

Integrate by parts to pick up

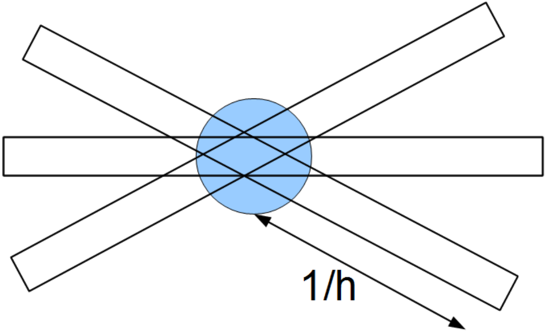
$$h\left|\frac{x-\gamma y}{|x-\gamma y|}-k\right|^{-1}$$

each time. Contribution is $O_N(h^{N\epsilon})$. These terms can be removed

$$T = \sum_{\text{lattice points } \xi_k} T_{\xi_k}$$

Now we have recovered the correct $L^2 \to L^\infty$ norm. What about other values of p.

Long overlapping tubes. Want to know what happens in an $O(1)$ region. Place a cut off there



No longer have a h^2 quasimode. Cut off makes it order h quasimode. So propagate for $O(1)$ time

Flowing for $O(1)$ time we have

$$u(x) = T^1 u$$

$$T^1 u = h^{-\frac{1}{2}} \int e^{\frac{i}{h}|x-y|} a(x,y) u(y) dy$$

with $a(x,y)$ supported in $\epsilon \leq |x-y| \leq 2\epsilon$. First split T^1 into K directions

$$T^1_\xi u = h^{-\frac{1}{2}} \int e^{\frac{i}{h}|x-y|} a(x,y) \zeta\left(K\left(\frac{x-y}{|x-y|} - \xi\right)\right) u(y) dy$$

$$T^1 = \sum_{i=1}^{K} T^1_{\xi_i}$$

Will study

$$\langle v, \sum_{j=1}^{K} T^1_{\xi_j} \rangle^N$$

$$\langle v, \sum_{i=j}^{K} T^1_{\xi_j} \rangle^N = \sum_{[j_1,\cdots,j_N]} \prod_{i=1}^{N} \langle v, T^1_{\xi_{i_j}} u \rangle$$

Most terms in the sum include approximately K distinct directions repeated equally. Will show that there is an improvement for spatially spread out terms. Let

$$T_{[j_1,\cdots,j_N]} u^{\otimes N} = \prod_{i=1}^{N} (T^1_{\xi_{j_i}} u)(x_j)$$

Symmeterize

$$T^{sym}_{[j_1,\cdots,j_N]} v(x_1, \cdots, x_N) = \frac{1}{(N!)^2} \sum_{\sigma,\pi \in S_N} \prod_{i=1}^{N} (T^1_{\xi_{j_{\sigma(i)}}} v)(x_{\pi(i)})$$

where S_N is the symmetric group of order N

We denote a point $X \in \mathbb{R}^{2N}$ as $X = (x_1, \dots, x_N)$. In this notation

$$T^{sym}_{[j_1,\cdots,j_N]} v(X) = \int K_{[j_1,\cdots,j_N]}(X,Y) v(Y) dY$$

where

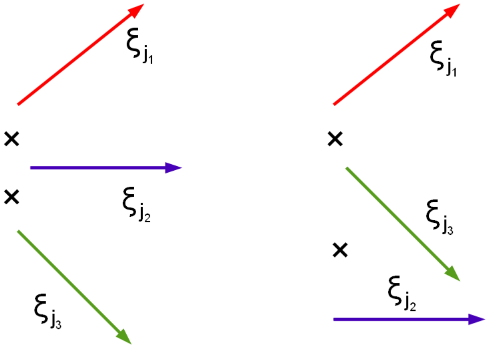
$$K_{j_1,\cdots,j_N]}(X,Y) = \frac{1}{(N!)^2} \sum_{\sigma,\pi \in S_N} K^{\sigma,\pi}_{[j_1,\cdots,j_N]}(X,Y)$$

$$K^{\sigma,\pi}_{[j_1,\cdots,j_N]}(X,Y) = \prod_{i=1}^{N} K^{\sigma}_{\xi_{j_{\sigma(i)}}}(x_{\pi(i)}, y_i)$$

$$K^{\sigma}_{\xi_{j_i}}(x,y) = e^{\frac{i}{h}|x-y|} a(x,y) \zeta\left(K \left(\frac{x-y}{|x-y|} - \xi_{j_i} \right) \right)$$

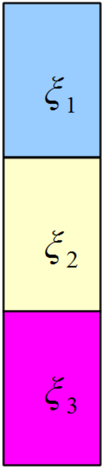
Fix π consider

$$\sum_{\sigma \in S_N} K^{\sigma,\pi}_{[j_1,\cdots,j_N]}(X,Y)$$



only get overlap if all $\xi_{j_{\sigma(i)}}$ are the same

How many overlaps can we have?

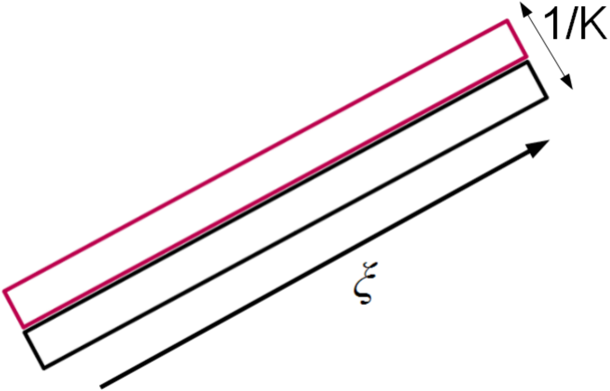


Each block can have $(N/K)!$ permutations within it. There are K blocks so

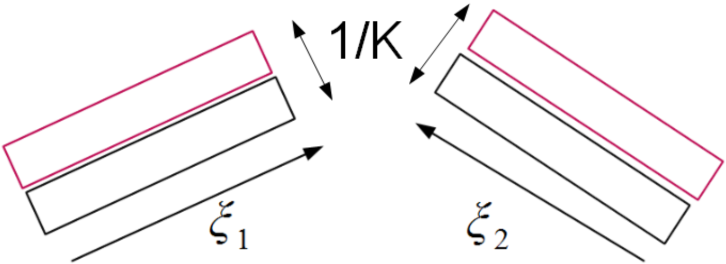
$$((N/K)!)^K \approx (N/K)^N$$

Compare with one $N!$ term to get an improvement of $(1/K^N)$. Since there are K^N ways of creating this kind of product this cancels out but gives the correct L^∞ estimate, we still have one copy of $(N!)$ left.

Look at directionally localized pieces. Shifting in short direction stops tubes from overlapping

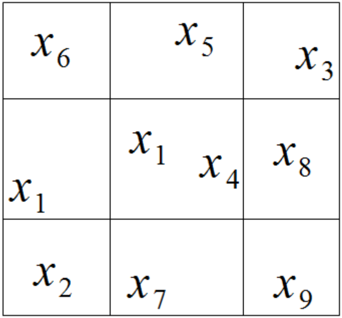


Look in two different directions.



Shifting in any direction must cause one direction to not overlap. Therefore in the product a shift in any direction will cause something to fail to overlap.

Divide $\mathbb{T}^2$ into boxes of size $1/K$



If $(x_1, \cdots x_N)$ is spread out among M boxes get and improvement of

$$\frac{((N/M)!)^N}{N!} \approx \frac{1}{M}$$

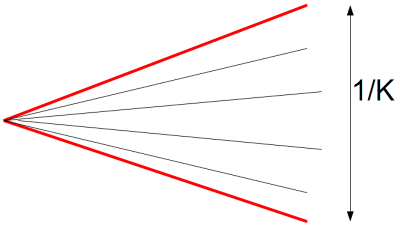
Each of these boxes is now an order hK quasimode. Repeat process by flowing for times $1/K$.

End result

As long at there is no loss this method will give

$$\|u\|_{L^p} \leq C \|u\|_{L^2}$$

for all $p < \infty$. Major possibility for loss is an inductive creep.



Need to treat clustered terms, do this inductively by further breaking them apart, need to watch for loss of constants.