L^p bounds for spectral projectors

Daniel Tataru

University of California, Berkeley

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This is joint work with Herbert Koch and Hart Smith

Estimates in $\mathbb{R}^d$

Main resolvent estimate:

$$\|(\Delta + \lambda^2 \pm i0)^{-1} f\|_{L^{p_d}} \lesssim \lambda^{-\frac{2}{d+1}} \|f\|_{L^{p'_d}}$$

Proof by complex interpolation:

$$(\Delta + \lambda^2 \pm i0)^{i\sigma} : L^2 \to L^2$$

$$(\Delta + \lambda^2 \pm i0)^{-\frac{d+1}{2}+i\sigma} : L^1 \to L^\infty$$

Kernel decay for $(\Delta + \lambda^2 \pm i0)^{-1}$: $\quad |x|^{-\frac{d-1}{2}}$.
$L^2 \to L^{p_d}$ formulation:

$$\|u\|_{L^{p_d}} \lesssim \lambda^{\frac{d-1}{2(d+1)}} (\lambda^{-1} \|(\Delta + \lambda^2) u\|_{L^2} + \|u\|_{L^2})$$

Spectral projector version:

$$\|P_{[\lambda,\lambda+1]} u\|_{L^{p_d}} \lesssim \lambda^{\frac{d-1}{2(d+1)}} \|u\|_{L^2}$$

Estimates in $\mathbb{R}^d$

Critical exponent:

$$p_d = \tfrac{2(d+1)}{d-1}$$

Full range of p's:

$$\|P_{[\lambda,\lambda+1]}u\|_{L^p(M)} \lesssim \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\,\|f\|_{L^2(M)}\,, \qquad p_d \leq p \leq \infty\,,$$

$$\|P_{[\lambda,\lambda+1]}u\|_{L^p(M)} \lesssim \lambda^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}\,\|f\|_{L^2(M)}\,, \qquad 2 \leq p \leq p_d\,, \qquad ,$$

Counterexamples:

- $p_d < p \leq \infty$: $\hat{u}$ bump in the annulus $|\xi| \in [\lambda, \lambda+1]$
- $2 \leq p \leq p_d$: $\hat{u}$ bump in a rectangle $1 \times \lambda^{-\frac{1}{2}}$ (Knapp counterexample)
  also all intermediate scales

Compact manifolds

(M, g) compact Riemannian manifold.

- g smooth: Sogge ..... Fourier Integral Operator parametrix, oscillatory integrals
- $g \in C^{1,1}$: Smith .... Wave packet parametrix
- $g \in C^s$, $0 < s < 2$: present talk ... estimates with losses.

Paradifferential calculus:

$$\Delta_g u_\lambda = \Delta_{g_{<\lambda^\sigma}} u_\lambda + error$$

- C^2 scale: $\delta x = \lambda^{\frac{s-2}{s+2}}$, $\sigma = \frac{2}{s+2}$, wave packet parametrix
- C^1 scale: $\delta x = \lambda^{s-1}$, $\sigma = 1$, energy propagation

C^1 metrics

Scales:

- C^2 scale: $\delta x = \lambda^{-\frac{1}{3}}$, $\sigma = \frac{2}{3}$
- wave packet size: $\lambda^{-\frac{2}{3}}$, angle: $\lambda^{-\frac{1}{3}}$

Enemies:

	angle	width	height	period	counterexample for
Wide	1	λ^{-1}	λ^{-1}	1	$p \geq \frac{2(d+2)}{d-1}$
Narrow	$\lambda^{-\frac{1}{3}}$	$\lambda^{-\frac{2}{3}}$	$\lambda^{-\frac{1}{3}}$	$\lambda^{-\frac{1}{3}}$	$p \leq \frac{2(d+2)}{d-1}$
Intermediate	$k\lambda^{-\frac{1}{3}}$	$k\lambda^{-\frac{2}{3}}$	$k^{-1}\lambda^{-\frac{1}{3}}$	$k\lambda^{-\frac{1}{3}}$	$p = \frac{2(d+2)}{d-1}$

Conjecture

The following bounds hold for C^1 metrics:

$$\|P_{[\lambda,\lambda+1]}u\|_{L^p(M)} \lesssim \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \|u\|_{L^2(M)}, \qquad \frac{2(d+2)}{d-1} \leq p \leq \infty,$$

$$\|P_{[\lambda,\lambda+1]}u\|_{L^p(M)} \lesssim \lambda^{\frac{2(d-1)}{3}(\frac{1}{2}-\frac{1}{p})} \|u\|_{L^2(M)}, \qquad 2 \leq p \leq \frac{2(d+2)}{d-1}.$$

Our result: $d = 2$

Theorem

The following bounds hold for C^1 metrics in dimension $d = 2$:

$$\|P_{[\lambda,\lambda+1]}u\|_{L^p(M)} \lesssim \lambda^{2(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \|u\|_{L^2(M)}, \qquad 8 < p \leq \infty,$$

$$\|P_{[\lambda,\lambda+1]}u\|_{L^p(M)} \lesssim \lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})} \|u\|_{L^2(M)}, \qquad 2 \leq p < 8.$$

$$\|P_{[\lambda,\lambda+1]}u\|_{L^8(M)} \lesssim (\log\lambda)^3 \lambda^{\frac{1}{4}} \|u\|_{L^2(M)}$$

- partial results for $d \geq 3$
- partial results for $d = 2$ in earlier paper
- log loss due to dyadic summations
- $p = 8$: enemies at all scales

Wave packets and bushes

Wave packets decomposition on C^2 scale $\delta t = \lambda^{-\frac{1}{3}}$

$$u = \sum a_T u_T$$

- wave packet scales: $\delta x = \lambda^{-\frac{2}{3}}$, $\delta \xi = \lambda^{\frac{2}{3}}$, $\delta \theta = \lambda^{-\frac{1}{3}}$
- Fourier coefficients a_T are nonconstant due to truncation errors
- dyadic decomposition with respect to size of a_T
- dyadic decomposition w.r. to time intervals on which $\|a_T'\| \ll a_T$.

k- Bushes (of pachets with comparable a_T):

$$\sum \chi_T \approx k$$

- Coherence time: $k^{-1}\lambda^{-\frac{1}{3}}$ (if focused)
- Minimal refocusing time $k\lambda^{-\frac{1}{3}}$ (worst case scenario)

Proof ideas:

Step 1: k-bush counting.

- Count $\delta t = k^{-1}\lambda^{-\frac{1}{3}}$ time slices containing k-bushes
- Count restricted to time intervals $\delta t = k\lambda^{-\frac{1}{3}}$
- No room for errors

Step 1: k-bush estimates.

- need to estimate all k-bushes in a $\delta t = k^{-1}\lambda^{-\frac{1}{3}}$ time slice
- bushes need not be focused
- overlapping can occur for bushes in different directions
- packets can belong to multiple bushes
- No room for errors

Bush counting

Main idea: Bushes on different slices are "almost orthogonal".
Assuming $a_T = 1$ define projectors

$$P_j = k^{-1} \sum u_T \langle \sum u_T, \cdot \rangle$$

Then

$$\|P_j S(t_j, t_l) P_l\|_{L^2 \to L^2} \lesssim k^{-1} \max\{\lambda^{-\frac{1}{3}} |t_j - t_l|^{-1}, \lambda^{\frac{1}{3}} |t_j - t_l|\}$$

Not enough for Cotlar's lemma.
a) Short time $|t_j - t_l| < \lambda^{-\frac{1}{3}}$ (below C^2 scale)

- can use C^2 parametrix
- count the number of packets though two bushes

b) Long time $|t_j - t_l| > \lambda^{-\frac{1}{3}}$ (above C^2 scale)

- cannot use C^2 parametrix
- use instead generalized coherent packets with $\delta x = |t_i - t_j|^2$, $\delta\xi = |t_i - t_j|\lambda$.

L^8 estimates for k-bushes

Main difficulty: k-bushes are not necessarily disjoint or focused.
Main idea: For $u = \sum u_T$ decompose with respect to angles

$$u^2 = \sum_{\angle(T,S) > k\lambda^{-\frac{1}{3}}} u_S u_T + \sum_{\angle(T,S) < k\lambda^{-\frac{1}{3}}} u_S u_T$$

a) Large angle interactions: use bilinear L^2 bound,

$$\|uv\|_{L^2} \lesssim \angle(u,v)^{-\frac{1}{2}} \|u_0\|_{L^2} \|v_0\|_{L^2}$$

and interpolate with the L^∞ bound (given by k).
b) Small angle interactions: by orthogonality it suffices to fix the position and direction. Then we are left with focused isolated bushes. For these use the L^6 Strichartz and interpolate with the L^∞ bound (again given by k).