The Stokes groupoids

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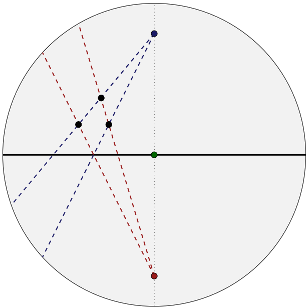
Fields Institute workshop on EDS and Lie theory, December 11, 2013



Based on arXiv:1305.7288 with Songhao Li and Brent Pym

Differential equations as connections

Any linear ODE, e.g.

$$\frac{d^2u}{dz^2} + \alpha \frac{du}{dz} + \beta u = 0,$$

can be viewed as a first order *system*: set $v = u'$ and then

$$\frac{d}{dz}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

This defines a flat connection

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ \beta & \alpha \end{pmatrix} dz,$$

so that the system is

$$\nabla f = 0.$$

Flat connections as representations

Flat connection on vector bundle E: for each vector field $\mathcal{V} \in \mathcal{T}_{\mathsf{X}}$,

$$\nabla_{\mathcal{V}} : \mathcal{E} \to \mathcal{E}$$

Curvature zero:

$$\nabla_{[\mathcal{V}_1, \mathcal{V}_2]} = [\nabla_{\mathcal{V}_1}, \nabla_{\mathcal{V}_2}].$$

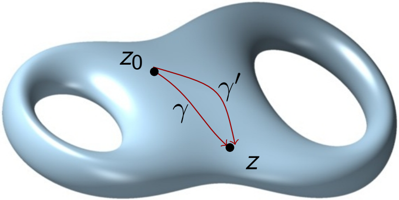
(E, ∇) is a *representation* of the Lie algebroid $\mathcal{T}_{\mathsf{X}}$.

Solving ODE

Fix an initial point z_0. Solving the equation along a path γ from z_0 to z gives an invertible matrix

$$\psi(z)$$

mapping an initial condition at z_0 to the value of the solution at z.



This is called a *fundamental solution* and its columns form a basis of solutions.

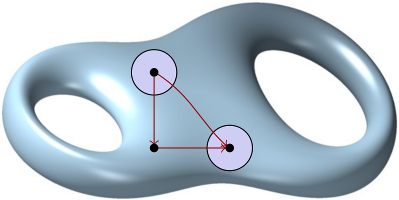
Also called *Parallel transport operator*, and depends only on the homotopy class of γ.

The fundamental groupoid

Define the **fundamental groupoid of** X:

$$\Pi_1(X) = \{\text{paths in } X\}/(\text{homotopies fixing endpoints})$$

- Product: concatenation of paths
- Identities: constant paths
- Inverses: reverse directions
- Manifold of dimension $2(\dim X)$



Parallel transport as a representation

The parallel transport gives a map

$$\Psi : \Pi_1(X) \to GL(n, \mathbb{C})$$

which is a **representation of** $\Pi_1(X)$:

$$\Psi(\gamma_1\gamma_2) = \Psi(\gamma_1)\Psi(\gamma_2)$$
$$\Psi(\gamma^{-1}) = \Psi(\gamma)^{-1}$$
$$\Psi(1_x) = 1$$

We call Ψ the **universal solution** of the system.

Riemann–Hilbert correspondence

Correspondence between differential equations, i.e. flat connections

$$\nabla : \Omega^0_{\mathsf{X}}(\mathcal{E}) \to \Omega^1_{\mathsf{X}}(\mathcal{E}),$$

and their solutions, i.e. parallel transport operators

$$\Psi(\gamma) : \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}.$$

Integration

$$\{\text{representations of } \mathcal{T}_{\mathsf{X}}\} \rightleftarrows \{\text{representations of } \Pi_1(\mathsf{X})\}$$

Differentiation

Main problem: singular ODE

A singular ODE leads to a singular (meromorphic) connection

$$\nabla = d + A(z)z^{-k}dz.$$

For example, the Airy equation $f'' = xf$ has connection

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ -x & 0 \end{pmatrix} dx,$$

and in the coordinate $z = x^{-1}$ near infinity,

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ -z & -z^2 \end{pmatrix} z^{-3}dz.$$

Singular ODE

Singular ODE have singular solutions:

$$f' = z^{-2} f \qquad f = Ce^{-1/z}$$

Formal power series solutions often have zero radius of convergence:

$$\nabla = d + \begin{pmatrix} -1 & z \\ 0 & 0 \end{pmatrix} z^{-2} dz$$

has solutions given by columns in the matrix

$$\psi = \begin{pmatrix} e^{-1/z} & \hat{f} \\ 0 & 1 \end{pmatrix},$$

where formally $\hat{f} = \sum_{n=0}^{\infty} n! z^{n+1}$.

Resummation

Borel summation/multi-summation: recover actual solutions from divergent series:

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \left(\tfrac{1}{n!z} \int_0^{\infty} t^n e^{-t/z} \, dt \right)$$
$$= \frac{1}{z} \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \right) e^{-t/z} \, dt$$

The auxiliary series may now converge.

Traditional solutions $\psi(z)$:

- multivalued
- not necessarily invertible
- essential singularities
- zero radius of convergence

Why? They are written on the *wrong space*. The correct space must be 2-dimensional analog of the fundamental groupoid.

The main idea

$\mathcal{T}_X(-D)$ as a Lie algebroid

View a meromorphic connection not as a representation of $\mathcal{T}_X$ with singularities on the divisor $D = k_1 \cdot p_1 + \cdots + k_n \cdot p_n$, but as a representation of the Lie algebroid

$$\begin{aligned} \mathcal{A} = \mathcal{T}_X(-D) &= \text{sheaf of vector fields vanishing at D} \\ &= \left\langle z^k \frac{\partial}{\partial z} \right\rangle \end{aligned}$$

$\mathcal{A}$ defines a vector bundle over X which serves as a replacement for the tangent bundle $\mathcal{T}_X$.

Definition: A Lie algebroid $(\mathcal{A}, [,], a)$ is a vector bundle $\mathcal{A}$ with a Lie bracket on its sections and a bracket-preserving bundle map

$$a : \mathcal{A} \to \mathcal{T}_{\mathsf{X}},$$

such that $[u, fv] = f[u, v] + (L_{a(u)}f)v$.

Definition: A representation of the Lie algebroid $\mathcal{A}$ is a vector bundle $\mathcal{E}$ with a flat $\mathcal{A}$-connection

$$\nabla : \mathcal{E} \to \mathcal{A}^* \otimes \mathcal{E}, \qquad \nabla(fs) = f\nabla s + (d_{\mathcal{A}}f)s.$$

For $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D}) = \langle z^k \partial_z \rangle$, we have $\mathcal{A}^* = \langle z^{-k} dz \rangle$, and so

$$\begin{aligned} \nabla &= d + A(z)(z^{-k}dz) \\ &= \left(z^k \partial_z + A(z)\right) z^{-k} dz, \end{aligned}$$

i.e. a meromorphic connection.

Definition: A Lie groupoid G over X is a manifold of arrows g between points of X.

- Each arrow g has source $s(g) \in X$ and target $t(g) \in X$. The maps $s, t : G \to X$ are surjective submersions.
- There is an associative composition of arrows

$$m : G_s \times_t G \to G.$$

- Each $x \in X$ has an identity $\mathrm{id}(x) \in G$; this gives an embedding $X \subset G$.
- Each arrow has an inverse.

Examples:

- The fundamental groupoid $\Pi_1(X)$.
- The pair groupoid $X \times X$, in which

$$(x, y) \cdot (y, z) = (x, z).$$

Given a Lie group K and a K-space X, the *action groupoid* $\mathrm{G} = K \times \mathrm{X}$ has structure maps

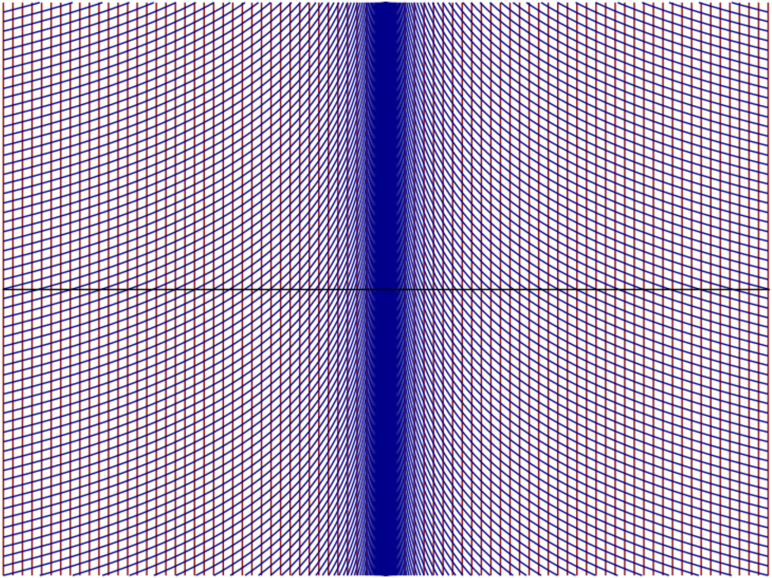
$$s(k, x) = x, \qquad t(k, x) = k \cdot x,$$

and obvious composition law.

For example, the action of $\mathbb{C}$ on $\mathbb{C}$ via

$$u \cdot z = e^u z$$

gives rise to a groupoid $\mathrm{G} = \mathbb{C} \times \mathbb{C}$ with the following structure:



Action groupoid for $\mathbb{C}$ action on $\mathbb{C}$ given by $u \cdot z = e^u z$.
Vertical lines are s-fibres and blue curves are t-fibres.

The Lie algebroid $\mathcal{A}$ of a Lie groupoid G over X is defined by:

$$\mathcal{A} = N(\mathrm{id}(\mathrm{X})) \cong \ker s_*|_{\mathrm{id}(\mathrm{X})}.$$

- Sections of $\mathcal{A}$ have unique extensions to right-invariant vector fields tangent to s-foliation $\mathcal{F}$. Thus $\mathcal{A}$ inherits a Lie bracket.
- t-projection defines the anchor a:

$$t_* : \mathcal{A} \to \mathcal{T}_{\mathrm{X}}.$$

Definition: A representation of a Lie groupoid G over X is a vector bundle $\mathcal{E} \to \mathrm{X}$ and an isomorphism

$$\Psi : s^*\mathcal{E} \to t^*\mathcal{E}, \quad \Psi_{gh} = \Psi_g \circ \Psi_h.$$

Integration: If $\mathcal{E}$ has a flat $\mathcal{A}$-connection, then $t^*\mathcal{E}$ has a *usual* flat connection along s-foliation $\mathcal{F}$.
$s^*\mathcal{E}$ is trivially flat along $\mathcal{F}$, and so the identification

$$s^*\mathcal{E}|_{\mathrm{id(X)}} = t^*\mathcal{E}|_{\mathrm{id(X)}}$$

may be extended uniquely to

$$\Psi : s^*\mathcal{E} \to t^*\mathcal{E},$$

as long as the s-fibres are simply connected.

In this way, we obtain an equivalence

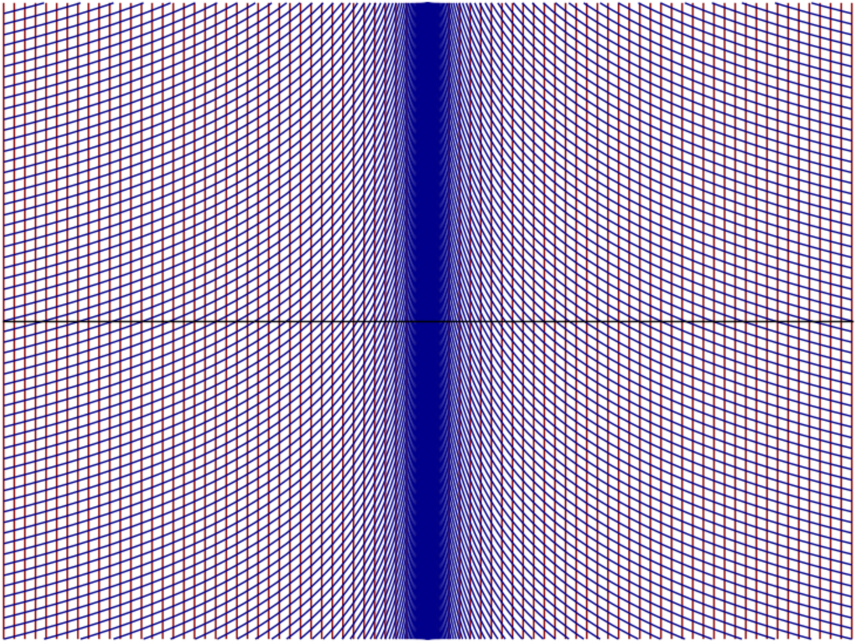
$$\mathbf{Rep}(\mathcal{A}) \leftrightarrow \mathbf{Rep}(\mathsf{G}),$$

using nothing more than the usual existence and uniqueness theorem for nonsingular ODEs.

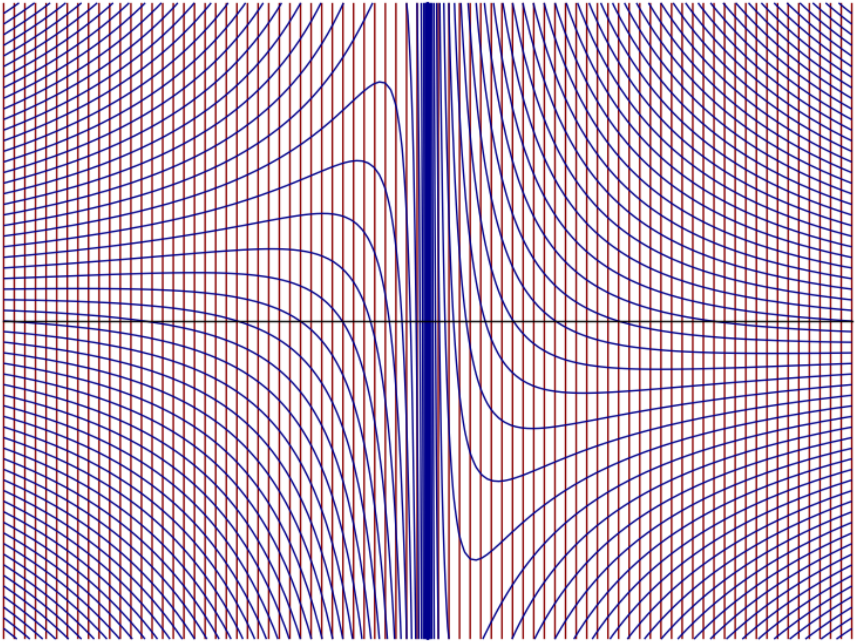
Example: $\mathrm{Sto}_k = \Pi_1(\mathbb{C}, k \cdot 0) = \mathbb{C} \times \mathbb{C}$ with

$$
\begin{aligned}
s(z,u) &= z \\
t(z,u) &= \exp(uz^{k-1})z \\
(z_2,u_2)\cdot(z_1,u_1) &= (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1).
\end{aligned}
$$

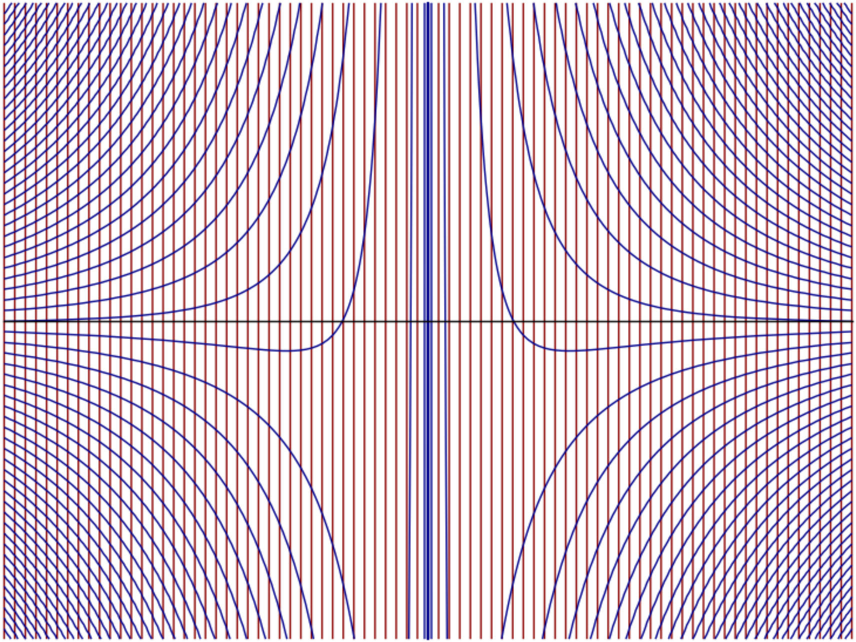
For $k = 1$, coincides with action groupoid, but for $k > 1$ not an action groupoid.



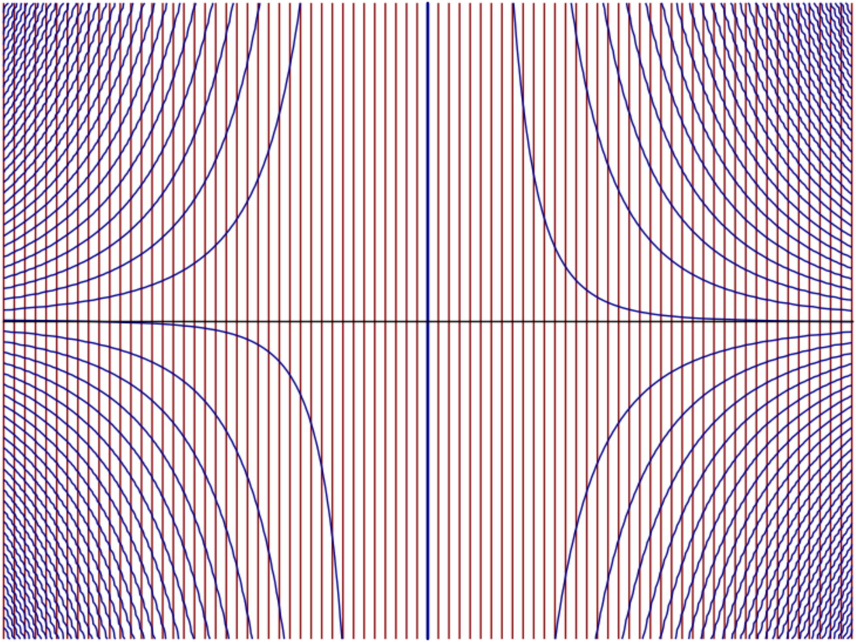
Sto$_1$ groupoid for 1st order poles on $\mathbb{C}$



Sto_2 groupoid for 2nd order poles on $\mathbb{C}$



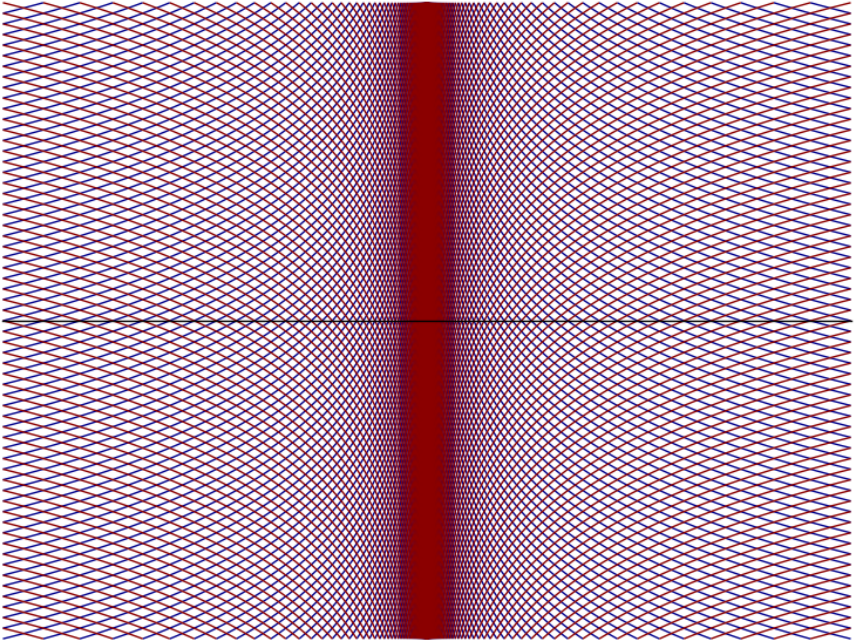
Sto_3 groupoid for 3rd order poles on $\mathbb{C}$



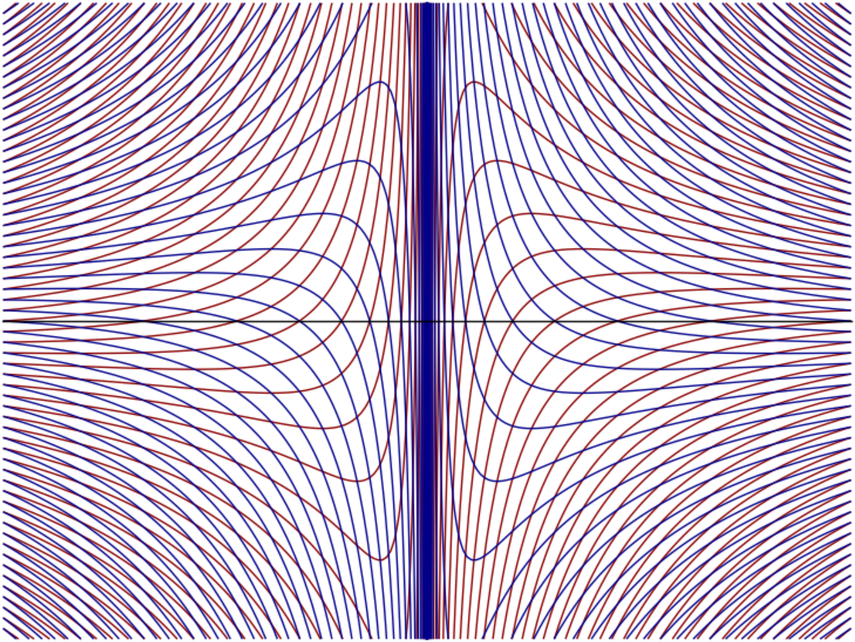
Sto_4 groupoid for 4th order poles on $\mathbb{C}$

We can write Sto_k more symmetrically:

$$s(z,u) = \exp(-\tfrac{1}{2}uz^{k-1})z$$
$$t(z,u) = \exp(\tfrac{1}{2}uz^{k-1})z$$



Sto$_1$ groupoid for 1st order poles on $\mathbb{C}$



Sto_2 groupoid for 2nd order poles on $\mathbb{C}$

Theorem: If ψ is a fundamental solution of $\nabla\psi = 0$, i.e. a flat basis of solutions, and if ∇ is meromorphic with poles bounded by D, then ψ may be

- multivalued
- non-invertible
- singular,

however

$$\boxed{\Psi = t^*\psi \circ s^*\psi^{-1}}$$

is single-valued, smooth and invertible on the Stokes groupoid.

Recall that the connection

$$\nabla = d + \begin{pmatrix} -1 & z \\ 0 & 0 \end{pmatrix} z^{-2} dz$$

has fundamental solution

$$\psi = \begin{pmatrix} e^{-1/z} & \widehat{f} \\ 0 & 1 \end{pmatrix},$$

$$\text{where formally } \widehat{f} = \sum_{n=0}^{\infty} n! z^{n+1}.$$

∇ is a representation of $\mathcal{T}_{\mathbb{C}}(-2 \cdot 0)$, and so the corresponding groupoid representation Ψ is defined on Sto_2. For convenience we use coordinates (z, μ) on the groupoid such that

$$s(z, \mu) = z, \quad t(z, \mu) = z(1 - z\mu)^{-1}.$$

$$\Psi = t^*\psi \circ s^*\psi^{-1} = t^* \begin{pmatrix} e^{-1/z} & \widehat{f} \\ & 1 \end{pmatrix} s^* \begin{pmatrix} e^{-1/z} & \widehat{f} \\ & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e^{-(1-z\mu)/z} & t^*\widehat{f} \\ & 1 \end{pmatrix} \begin{pmatrix} e^{1/z} & -s^*\widehat{f} \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\mu} & t^*\widehat{f} - e^{\mu} s^*\widehat{f} \\ & 1 \end{pmatrix}$$

But we know a priori this converges on the groupoid:

Indeed, using $\widehat{f} = \sum_{n=0}^{\infty} n!\, z^{n+1}$,

$$t^* \widehat{f} - e^{\mu} s^* \widehat{f} = -\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^{i+1} \mu^{i+j+1}}{(i+1)(i+2)\cdots(i+j+1)},$$

which is a convergent power series in two variables for the representation Ψ.