DYNAMICS OF METRICS AND SCALING ENTROPY OF THE GROUP ACTIONS.

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3.For each $\epsilon > 0$ the σ-field $\mathfrak{A}$ generated by the set of all balls of radius ϵ.

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(equivalent)
4. For each $\epsilon > 0$ a set $X_\epsilon : \mu X_\epsilon > 1 - \epsilon$ is precompact.
5. Lemma: If ρ satisfies to metric axioms as measurable functions w.r.t. $\mu \times \mu, \mu \times \mu \times \mu$ then there exists a set of measure 1 and true metric on it which is a.e. coincide with ρ.

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Cone $K_{(X,\mu)}$ of all admissible metrics on the standard measure spaces.
Norm and topology on the space of admissible metrics: L^1-norm and M-norm. Compactness in and entropy.

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Lemma (Equivalence of topology)

Let ρ_1 ρ_2 — two admissible metrics on (X, μ). Then for each $\epsilon > 0$ there a measurable set $K \subset X, \mu(K) > 1 - \epsilon$ s.t. topology generated by metrics ρ_1 and ρ_2 on K are coincided.

Classification of admissible triples

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$(X, \mu, \rho) \sim (X', \mu', \rho')$ iff

$$\exists T : X \to X'; \quad T_* \mu = \mu' \quad \rho'(Tx, Ty) = \rho(x, y)$$

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Theorem

(Gromov-V.) Define the map:

$$F_\rho : X^\infty \times X^\infty \to M_\infty(\mathbb{R}) \quad F_\rho(\{x_i\}_i, \{y_j\}_j) = \{\rho(x_i, y_j)\}_{i,j},$$

define $X^\infty \times X^\infty$, a product (Bernoulli) measure $\mu^\infty \times \mu^\infty \equiv \mu^{2\infty}$. Then the image of measure $\mu^{2\infty}$ under the map: $F_{\rho_*}(\mu^{2\infty}) \equiv D_\rho$, which called MATRIX DISTRIBUTION OF THE METRIC ρ with respect to measure μ
is the **complete invariant of the equivalence of metric triple.**

Roughly speaking the random metric on $\mathbb{N}$ is an invariant of metric on continuous space.

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either
the standard (Lebesgue) measure space with continuous measure and a metric of Universal Urysohn space, — if we fix a measure;
or
Universal Urysohn space with non-degenerated (=all nonempty open set have positive measure) continuous measure, — if we fixed a generic metric.

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Here "generic" means the element of everywhere dense G_δ-set of the space of all admissible triples with respect to natural topology.

Virtually continuous functions

. Let $f(\cdot,\cdot)$ be a measurable function of two variables. Then Luzin's theorem analogue (continuity on the product $X' \times Y'$ of sets of measure $> 1-\epsilon$ with respect to given metric $\rho[(x_1,y_1),(x_2,y_2)] = \rho_X(x_1,x_2) + \rho_Y(y_1,y_2))$ is not in general true. This leads to the following key notion of this work.

Definition

Measurable function $f(\cdot,\cdot)$ on the product $(X,\mu) \times (Y,\nu)$ of standard spaces is called *virtually continuous*, if for any $\varepsilon > 0$ there exist sets $X' \subset X, Y' \subset Y$, each of which having measure $1-\epsilon$, and admissible semi-metrics ρ_X, ρ_Y on X', Y' respectively such that function f is continuous on $(X' \times Y', \rho_X \times \rho_Y)$. virtual functions of several variables are defined in the same way.

Main theorem: Virtually continuous function can be integrated over special kind of singular (with respect to product measure) measures. For example over "diagonal" or sub-manifolds. .

Theorem

Any admissible metric is virtually continuous function.

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The first invariant is — scaling entropy — generalization of Kolmogorov entropy.

Scaling entropy

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Define a sequence of positive numbers

$$H(X, \rho_n, \epsilon), \quad n = 1 \ldots,$$

as ϵ-entropy of the triple (X, μ, ρ_n), .., logarithm of minimal number of points in the ϵ-net over all measurable compact sets $X_\epsilon \subset X$, of measure $> 1 - \epsilon$ with respect to metrics ρ_n.

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Definition

Scaling sequence $\{c_{n,\epsilon}\}$ is the sequence for which the following condition is true

$$0 < \liminf \frac{H(X, \rho_n, \epsilon)}{c_{n,\epsilon}} \leq \limsup \frac{H(X, \rho_n, \epsilon)}{c_{n,\epsilon}} < \infty$$

Two scaling sequences for given metric are equivalent (ratio tends to 1 on infinity)

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Connection with theory of filtrations: 3.RWRS=Random walk on Random Scenery:
$c_n \sim (ln|G_n|)^k$ for locally finite groups and for $\mathbb{R}$ ($\mathbb{Z}$) — (Conjecturally horocycle-flow).

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Connection: A.Kirillov-A.Kushnirenko sequential entropy, S.Ferenci, A.Katok-J-P.Thouvenot (etc.) (V.St.Petersburg Math.Journ. 2011, N1).

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Graded graph, Numeration, Substitutions.
Pascal automorphism:
$P : \prod_{i=1}^{\infty} \{0; 1\} \circlearrowright$

$$\underbrace{0, \ldots, 0}_{m_1} \underbrace{1, \ldots, 1}_{k_1} ** = 0^{m_1} 1^{k_1} **.$$

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Pascal automorphism can be written by the following formula:

$$x \mapsto Px; \quad P(0^m1^k\mathbf{10}**) = 1^k0^m\mathbf{01}**, \quad m,k = 0,1\ldots.$$

The automorphism P^{-1} in a similar form:

$$P^{-1}(1^k0^m\mathbf{01}**) = 0^m1^k\mathbf{10}**, \quad m,k = 0,1\ldots.$$

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Hint:Pascal has the same orbits as the action o infinite symmetric group by permutations of the coordinates. The scaling sequence for entropy of that action is $\frac{\ln n!}{\ln n}$.