ARNOLD STABILITY of TIME-OSCILLATING FLOWS

Legacy of Vladimir Arnold

Fields Institute, November, 2014

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...

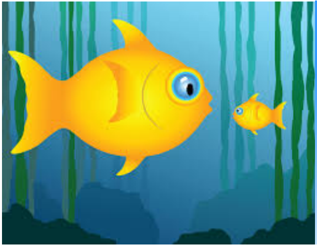
December 2, 2014

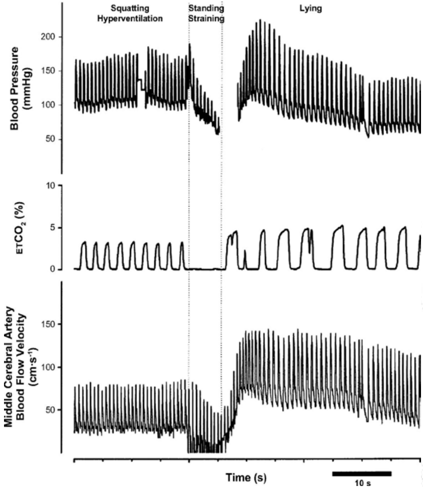
I like this great photo of Vladimir Igorevich

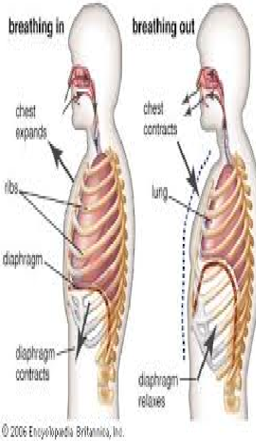
- **Oscillating flows appear in various applications**: geophysics, coastal engineering, self-swimming, medicine, machinery ... One can say that oscillating flows are the most important in applied hydrodynamics ...
- The flow oscillations can be caused by various factors such as oscillating boundaries, surface waves, acoustic waves, MHD waves, *etc.*
- Our aim is to present **asymptotic/averaging models for oscillating fluid flows** with the use of the **multi-scale (two-timing) method**.

- We consider **relatively 'weak' averaged flows**, interacting with the flow oscillations.
- The distinctive property of the averaged flows is: they possess an additional advection with the **drift velocity**.
- All our consideration is Eulerian. The drift velocity is Lagrangian characteristic of a flow, however in our consideration it naturally appears in an Eulerian procedure.
- The relations to the **Stokes drift**, **Langmuir circulations**, acoustics, and MHD dynamo are discussed.

- Our models represent examples of **Hamiltonian systems** and interesting areas of exploiting of **Arnold's ideas in Hydrodynamics.**
- The averaged equations and boundary conditions possess the **energy-type integral**, which allows us to consider some 'energy-related' results.
- We have derived a number of results such as the **energy variational principle**, the **second variation of energy**, and some **Arnold-type stability criteria** for averaged flows.




Squatting
Hyperventilation
Standing
Straining
Lying
Blood Pressure (mmHg)
200
150
100
50
$ETCO_2$ (%)
10
5
0
Middle Cerebral Artery Blood Flow Velocity ($cm \cdot s^{-1}$)
150
100
50
Time (s)
10 s


breathing in
breathing out
chest expands
ribs
diaphragm
diaphragm contracts
chest contracts
lung
diaphragm relaxes
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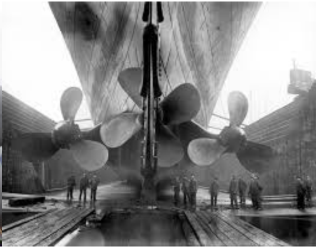
Oscillating Flows in turbine-applications: Slide 4C



Drift motion brings impressive income: Slide 4D

Langmuir Circulations in a lake: Slide 4E

A homogeneous inviscid incompressible fluid in a 3D domain Q with oscillating boundary ∂Q. Velocity $\mathbf{u}^\dagger = \mathbf{u}^\dagger(\mathbf{x}^\dagger, t^\dagger)$, vorticity $\boldsymbol{\omega}^\dagger \equiv \nabla^\dagger \times \mathbf{u}^\dagger$

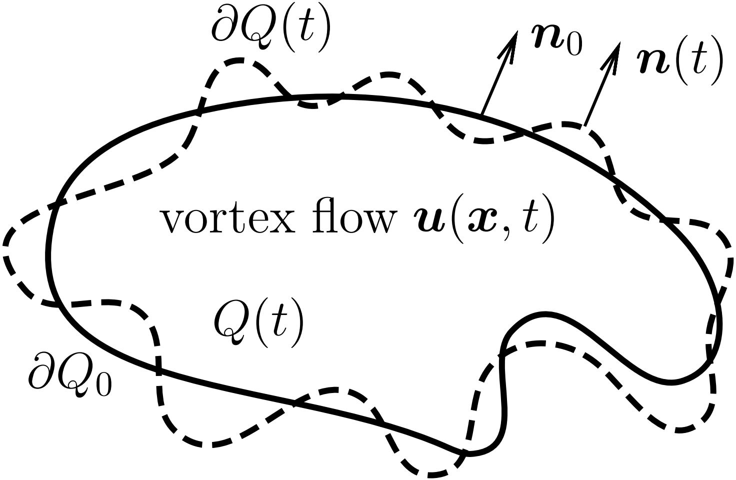
$$\frac{\partial \boldsymbol{\omega}^\dagger}{\partial t^\dagger} + [\boldsymbol{\omega}^\dagger, \mathbf{u}^\dagger]^\dagger = 0, \quad \nabla^\dagger \cdot \mathbf{u}^\dagger = 0$$

where 'dags' mark dimensional variables, and square brackets stand for the commutator. The boundary condition at ∂Q is

$$dF^\dagger/dt^\dagger = 0 \quad \text{at} \quad F^\dagger(\mathbf{x}^\dagger, t^\dagger) = 0$$

The characteristic scales of velocity, length, and two additional time-scales

$$U^\dagger, \quad L^\dagger, \quad T^\dagger_{\text{fast}}, \quad T^\dagger_{\text{slow}}$$



Oscillating flow domain $Q(t)$

Two independent dimensionless parameters

$$T_{\text{fast}} \equiv T^{\dagger}_{\text{fast}}/T^{\dagger}, \quad T_{\text{slow}} \equiv T^{\dagger}_{\text{slow}}/T^{\dagger}, \quad \text{where} \quad T^{\dagger} \equiv L^{\dagger}/U^{\dagger}$$

T_{fast} – the given period of oscillations, the frequency of oscillations

$$\sigma^{\dagger} \equiv 1/T^{\dagger}_{\text{fast}}, \quad \sigma \equiv T^{\dagger}/T^{\dagger}_{\text{fast}}$$

The dimensionless independent variables

$$\mathbf{x} \equiv \mathbf{x}^{\dagger}/L^{\dagger}, \quad t \equiv t^{\dagger}/T^{\dagger}$$

The dimensionless 'fast time' τ and 'slow time' s:

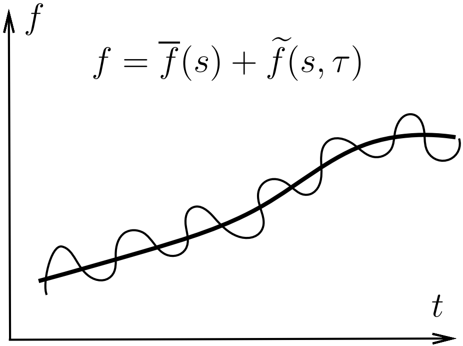
$$\tau \equiv t/T_{\text{fast}} = \sigma t, \qquad s \equiv t/T_{\text{slow}} \equiv St, \qquad S \equiv T^{\dagger}/T^{\dagger}_{\text{slow}}$$

Attention! $T^{\dagger}_{\text{slow}}$ is NOT given! It is unknown!

We consider the oscillatory solutions

$$\mathbf{u}^{\dagger} = AU^{\dagger}\mathbf{u}(\mathbf{x}, s, \tau)$$

where τ-dependence is 2π-periodic, s-dependence is general, A – the dimensionless amplitude of velocity.



Dimensionless variables and the chain rule give

$$\left(\frac{\partial}{\partial\tau}+\frac{S}{\sigma}\frac{\partial}{\partial s}\right)\boldsymbol{\omega}+\frac{A}{\sigma}[\boldsymbol{\omega},\mathbf{u}]=0$$

where s and τ are still mutually dependent variables.

An auxiliary assumption: *we operate with s and τ as mutually independent variables*; justification of it often can be given *a posteriori* by the estimation of the errors/residuals in the equation, rewritten back to the original variable t.

Two independent small parameters

- In the two-timing method the basic small parameter is

$$T_{\text{slow}}/T_{\text{fast}} = S/\sigma$$

- The term $\partial\boldsymbol{\omega}/\partial\tau$ must be dominating, in order to form an evolution equation. Hence, generally, we take two independent small parameters $\varepsilon_1, \varepsilon_2$ as:

$$\boldsymbol{\omega}_\tau + \varepsilon_1 \boldsymbol{\omega}_s + \varepsilon_2[\boldsymbol{\omega}, \mathbf{u}] = 0;\ \varepsilon_1 \equiv \frac{S}{\sigma} \ll 1,\ \varepsilon_2 \equiv \frac{A}{\sigma} \leq 1$$

  • ε_1 is ratio of two characteristic time scales;
  • ε_2 is the ratio of amplitude over frequency. Note: the amplitude itself can be huge!

- Asymptotic solutions correspond to the limit $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.

- There are infinitely many asymptotic pathes $(\varepsilon_1, \varepsilon_2) \to (0,0)$. QUESTION: Is the number of different asymptotic solutions also infinite?
- We accept that the *distinguished limit* is given by such a path $(\varepsilon_1, \varepsilon_2) \to (0,0)$ that allows us to build a self-consistent asymptotic procedure, leading to the finite/valid solution in any approximation.
- ANSWER: By the method of trial and errors one can find that there are only two pathes, which allow to build such solutions:

$$\varepsilon_1 = \varepsilon_2 \equiv \varepsilon : \quad \boldsymbol{\omega}_\tau + \varepsilon \boldsymbol{\omega}_s + \varepsilon[\boldsymbol{\omega}, \mathbf{u}] = 0$$
$$\varepsilon_1 = \varepsilon_2^2 \equiv \varepsilon^2 : \quad \boldsymbol{\omega}_\tau + \varepsilon^2 \boldsymbol{\omega}_s + \varepsilon[\boldsymbol{\omega}, \mathbf{u}] = 0$$

  The second case leads to the Weak Vortex Dynamics (WVD).
- Any systematic procedure of finding all possible distinguished limits is unknown: it can be classified as an experimental mathematics (Arnold). This is why pure mathematicians do not like this research area.

Any function $f = f(\mathbf{x}, s, \tau)$ in this paper is:

- $f = O(1)$ and all $\mathbf{x}$-, s-, and τ-derivatives of f are also $O(1)$.
- $f(\mathbf{x}, s, \tau) = f(\mathbf{x}, s, \tau + 2\pi)$
- The *averaging operation* is

$$\langle f \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0+2\pi} f(\mathbf{x}, s, \tau)\, d\tau, \qquad \forall\ \tau_0$$

- The *tilde-functions* (or purely oscillating functions) is such that

$$\widetilde{f}(\mathbf{x}, s, \tau) = \widetilde{f}(\mathbf{x}, s, \tau + 2\pi), \quad \text{with} \quad \langle \widetilde{f} \rangle = 0,$$

- The class of *bar-functions* is defined as

$$\overline{f}: \quad \overline{f}_\tau \equiv 0, \quad \overline{f}(\mathbf{x}, s) = \langle \overline{f}(\mathbf{x}, s) \rangle$$

- *The tilde-integration* keeps the result in the tilde-class:

$$\widetilde{f}^\tau \equiv \int_0^\tau \widetilde{f}(\mathbf{x}, s, \sigma)\, d\sigma - \frac{1}{2\pi} \int_0^{2\pi} \Big(\int_0^\mu \widetilde{f}(\mathbf{x}, s, \sigma)\, d\sigma \Big)\, d\mu.$$

We are looking for the solutions as regular series

$$\boldsymbol{\omega}_\tau + \varepsilon[\boldsymbol{\omega}, \mathbf{u}] + \varepsilon^2 \boldsymbol{\omega}_s = 0, \quad \varepsilon \to 0$$

$$(\boldsymbol{\omega}, \mathbf{u}) = \sum_{k=0}^{\infty} \varepsilon^k (\boldsymbol{\omega}_k, \mathbf{u}_k), \quad k = 0, 1, 2, \ldots$$

Our choice: the leading terms for the mean vorticity and mean velocity vanishes:

$$\overline{\boldsymbol{\omega}}_0 \equiv 0 \quad \overline{\mathbf{u}}_0 \equiv 0$$

It means that relatively weak vorticity develops on the background of a wave motion.
The zero approximation is $\widetilde{\boldsymbol{\omega}}_{0\tau} = 0$, its unique solution (within the tilde-class) is $\widetilde{\boldsymbol{\omega}}_0 \equiv 0$. Then full vorticity vanishes

$$\boldsymbol{\omega}_0 \equiv 0$$

Hence the flow in zero approximation is purely oscillatory and potential.

The equation of second approximation is

$$\widetilde{\boldsymbol{\omega}}_{2\tau} = -[\overline{\boldsymbol{\omega}}_1, \widetilde{\mathbf{u}}_0]$$

which yields

$$\widetilde{\boldsymbol{\omega}}_2 = [\widetilde{\mathbf{u}}_0^\tau, \overline{\boldsymbol{\omega}}_1], \quad \overline{\boldsymbol{\omega}}_2 = \ \textbf{?}$$

The equation of third approximation is

$$\widetilde{\boldsymbol{\omega}}_{3\tau} + \overline{\boldsymbol{\omega}}_{1s} + [\boldsymbol{\omega}_2, \widetilde{\mathbf{u}}_0] + [\overline{\boldsymbol{\omega}}_1, \mathbf{u}_1] = 0$$

Its bar-part is

$$\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_1, \overline{\mathbf{u}}_1] + \langle [\widetilde{\boldsymbol{\omega}}_2, \widetilde{\mathbf{u}}_0] \rangle = 0$$

which can be transformed to:

$$\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_1, \overline{\mathbf{u}}_1 + \overline{\mathbf{V}}_0] = 0$$

$$\overline{\mathbf{V}}_0(\mathbf{x}) \equiv \frac{1}{2} \langle [\widetilde{\mathbf{u}}_0, \widetilde{\mathbf{u}}_0^\tau] \rangle$$

After the dropping of subscripts and bars we get the WVD model

$$\boldsymbol{\omega}_s + [\boldsymbol{\omega}, \mathbf{w}] = 0, \quad \text{where} \quad \mathbf{w} \equiv \mathbf{u} + \overline{\mathbf{V}}_0$$

which shows that the averaged vorticity is ‘frozen’ into the ‘velocity+drift’.

The oscillating boundary is given by an exact expression

$$F(\mathbf{x}, t) = \overline{F}_0(\mathbf{x}, s) + \varepsilon \widetilde{F}_1(\mathbf{x}, s, \tau) = 0$$

The same steps applied to $dF/dt = 0$ lead to

$$\overline{F}_{0s} + \mathbf{w} \cdot \nabla \overline{F}_0 = 0, \quad \mathbf{w} \equiv \mathbf{u} + \overline{\mathbf{V}}_0$$

When $\overline{F}_{0s} = 0$, it gives the averaged no-leak condition:

$$\mathbf{w} \cdot \overline{\mathbf{n}}_0 = 0 \quad \text{or} \quad \mathbf{u} \cdot \overline{\mathbf{n}}_0 = -\overline{\mathbf{V}}_0 \cdot \overline{\mathbf{n}}_0 \quad \text{at} \quad \overline{F}_0(\mathbf{x}) = 0$$

The boundary conditions are valid not at the real boundary, but at its averaged position.

- The advection of an averaged vector-field is

$$\boldsymbol{\omega}_s + [\boldsymbol{\omega}, (\mathbf{u} + \overline{\mathbf{V}}_0)] = 0$$

which shows that the averaged vorticity is 'frozen' into the 'velocity+drift'.

- The advection of an averaged scalar-field appears as

$$\overline{F}_{0s} + (\mathbf{u} + \overline{\mathbf{V}}_0) \cdot \nabla \overline{F}_0 = 0$$

- One can see that the Lagrangian property (the drift velocity $\overline{\mathbf{V}}_0$) naturally appears in the Eulerian description after the averaging over oscillations.

Hence the problem for the purely oscillating boundaries ∂Q can be formulated as

$$\mathbf{u}_s + (\mathbf{u}\cdot\nabla)\mathbf{u} + \boldsymbol{\omega}\times\overline{\mathbf{V}}_0 = -\nabla p, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad Q_0$$

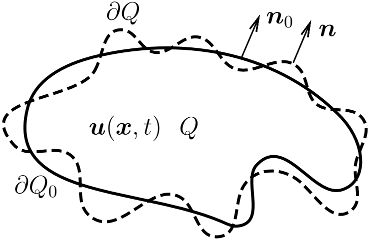
With the leak boundary condition:

$$\mathbf{u}\cdot\overline{\mathbf{n}}_0 = -\overline{\mathbf{V}}_0\cdot\overline{\mathbf{n}}_0 \quad \text{at} \quad \partial Q_0$$

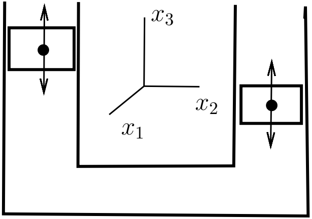
The boundary conditions are valid not at the real boundary, but at its averaged position.
The drift velocity is to be recovered from an independent problem

$$\overline{\mathbf{V}}_0(\mathbf{x}) \equiv \frac{1}{2}\langle[\widetilde{\mathbf{u}}_0, \widetilde{\mathbf{u}}_0^{\tau}]\rangle$$

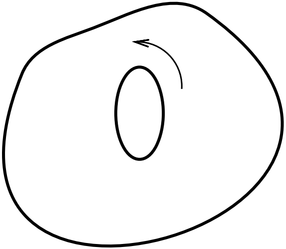
where $\widetilde{\mathbf{u}}_0$ represents the solution of previous approximation $\widetilde{\mathbf{u}}_{0\tau} = -\nabla p_0$ and $\operatorname{div}\widetilde{\mathbf{u}}_0 = 0$ with appropriate boundary conditions.



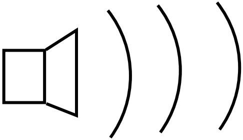
(a) flexible oscillating walls



(b) Oscillating pistons in U - tube.



(c) Rotating rigid body



(d) Acoustic wave

The *dimensional* solution for a plane potential travelling gravity wave is

$$\mathbf{u}_0^\dagger = U^\dagger \widetilde{\mathbf{u}}_0, \quad \widetilde{\mathbf{u}}_0 = \exp(k^\dagger z^\dagger) \begin{pmatrix} \cos(k^\dagger x^\dagger - \tau) \\ \sin(k^\dagger x^\dagger - \tau) \end{pmatrix}$$

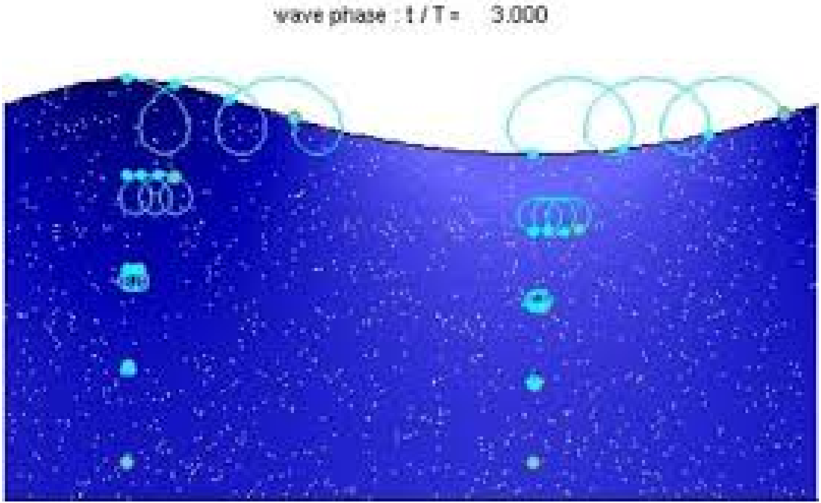
$U^\dagger = k^\dagger g^\dagger a^\dagger / \sigma^\dagger$ where $\sigma^\dagger$, $a^\dagger$, and $g^\dagger$ are dimensional frequency, spatial wave amplitude, and gravity. Then

$$\widetilde{\mathbf{u}}_0 = e^z \begin{pmatrix} \cos(x - \tau) \\ \sin(x - \tau) \end{pmatrix}, \quad \overline{\mathbf{V}}_0 = e^{2z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The dimensional version

$$\overline{\mathbf{V}}_0^\dagger = \frac{U^2 k^\dagger}{\sigma^\dagger} e^{2k^\dagger z^\dagger} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which agrees with the classical expression for the drift velocity.



Drift motion of a material particle due to a surface wave.

Transactionally invariant averaged flows + plane potential travelling gravity wave; (x, y, z) be such that $\overline{\mathbf{V}}_0 = (U, 0, 0)$, $U = e^{2z}$, $\overline{\mathbf{u}}_1 = (u, v, w)$. Then the component form of (1) is

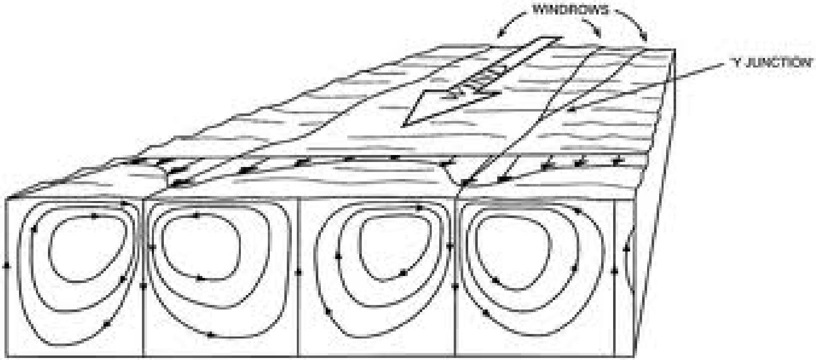
$$\begin{aligned} & u_s + vu_y + wu_z = 0 \\ & v_s + uv_y + wv_z - Uu_y = -\overline{p}_y \\ & w_s + vw_y + ww_z - Uu_z = -\overline{p}_z \\ & v_y + w_z = 0 \end{aligned}$$

it can be rewritten as

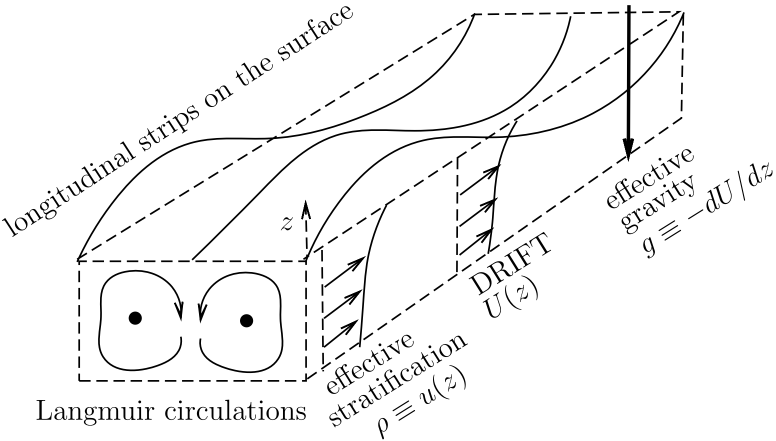
$$\begin{aligned} & v_s + vv_y + wv_z = -\overline{P}_y - \rho\Phi_y \\ & w_s + vw_y + ww_z = -\overline{P}_z - \rho\Phi_z \\ & v_y + w_z = 0 \\ & \rho_s + u\rho_x + v\rho_y = 0 \end{aligned}$$

where $\rho \equiv u$, $\Phi \equiv U = e^{2z}$, and $\overline{P}$ is a new modified pressure. It is equivalent to an incompressible stratified fluid.

The effective 'gravity field' $\mathbf{g} = -\nabla\Phi = (0, 0, -2e^{2z})$ is non-homogeneous. Nevertheless longitudinal vortices appear as a 'Taylor instability' of an inversely stratified equilibrium which corresponds to $(u, v, w) = (u(z), 0, 0)$ with any increasing function $u(z) \equiv \rho(z)$.



Qualitative pattern of Langmuir circulations



Generation of Langmuir circulations is equivalent to the Rayleigh-Taylor instability of a fluid with an inverse density stratification.

Surprisingly, the averaged equations for acoustics are the same as for incompressible fluid

$$\mathbf{u}_s + (\mathbf{u}\cdot\nabla)\mathbf{u} + \boldsymbol{\omega}\times\overline{\mathbf{V}}_0 = -\nabla p, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad Q_0$$

$$\mathbf{u}\cdot\overline{\mathbf{n}}_0 = -\overline{\mathbf{V}}_0\cdot\overline{\mathbf{n}}_0 \quad \text{at} \quad \partial Q_0$$

$$\overline{\mathbf{V}}_0 \equiv \frac{1}{2}\langle[\widetilde{\mathbf{u}}_0, \widetilde{\mathbf{u}}_0^{\tau}]\rangle$$

The difference is: $\overline{\mathbf{V}}_0$ can be NOT solenoidal!
Also one can suggest that by a proper configuration of acoustic wave, one can obtain almost ANY field of $\overline{\mathbf{V}}_0(\mathbf{x})$

For the incompressible MHD the averaged equations are

$$\boldsymbol{\omega}_s + [\boldsymbol{\omega}, \mathbf{u} + \mathbf{V}] - [\mathbf{j}, \mathbf{h}] = 0, \quad \mathbf{j} = \operatorname{curl} \mathbf{h}$$
$$\mathbf{h}_s + [\mathbf{h}, \mathbf{u} + \mathbf{V}] = 0, \quad \operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \mathbf{h} = 0$$
$$\overline{\mathbf{V}}_0 \equiv \langle [\widetilde{\mathbf{u}}_0, \widetilde{\mathbf{u}}_0^{\tau}] \rangle / 2$$

It can be derived by similar consideration. This system of equations is studying now for so-called MHD Stokes drift dynamo. The question about a general MHD-dynamo is completely open.

The ‘energy’ integral for the averaged WVD motion can be written as:

$$E = E(s) = \frac{1}{2}\int_Q (\mathbf{u} + \overline{\mathbf{V}}_0)^2 d\mathbf{x} = \text{const}, \qquad d\mathbf{x} \equiv dx_1 dx_2 dx_3$$

One can show that its s-derivative can be written as

$$\frac{dE}{ds} = -\int_Q \left(p + \frac{\mathbf{u}^2}{2}\right)(\mathbf{u} + \overline{\mathbf{V}}_0) \cdot \overline{\mathbf{n}}_0 d\mathbf{x} = 0$$

which is zero due to the BC.

According to (1) vorticity is 'frozen' into $\mathbf{u} + \overline{\mathbf{V}}_0$. It allows us to use the slightly generalized **Arnold isovorticity condition in its differential form**

$$\mathbf{u}_\theta = \mathbf{f} \times \boldsymbol{\omega} + \nabla\alpha, \quad \operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{f} = 0; \quad \text{in} \quad Q_0$$
$$(\mathbf{u} + \overline{\mathbf{V}}_0)\cdot\overline{\mathbf{n}}_0 = 0, \quad \mathbf{f}\cdot\overline{\mathbf{n}}_0 = 0 \quad \text{at} \quad \partial Q_0$$

where $\mathbf{u}(\mathbf{x},\theta)$ is the unknown function, $\mathbf{f} = \mathbf{f}(\mathbf{x},\theta)$ is an arbitrary given solenoidal function, θ is a scalar parameter along an isovortical trajectory, subscript θ stands for the related partial derivative. Function $\alpha(\mathbf{x},\theta)$ is to be determined from the condition $\operatorname{div}\mathbf{u} = 0$. The initial data at $\theta = 0$ for $\mathbf{u}(\mathbf{x},\theta)$ (1) correspond to a steady flow

$$\mathbf{u}(\mathbf{x},0) = \mathbf{U}(\mathbf{x}), \quad \boldsymbol{\omega}(\mathbf{x},0) = \boldsymbol{\Omega}(\mathbf{x})$$

where $\mathbf{U}(\mathbf{x})$ and $\boldsymbol{\Omega}(\mathbf{x})$ represent the steady solutions $(\partial/\partial s = 0)$ with 'no-leak' boundary conditions.

Differentiation of E with respect θ produces the zero of first variation

$$E_\theta\Big|_{\theta=0} = \int_{Q_0} \mathbf{f}\,(\boldsymbol{\Omega}\times\mathbf{W})\,d\mathbf{x} = 0, \quad \mathbf{W}\equiv\mathbf{U}+\overline{\mathbf{V}}_0$$

which vanishes for any function **f** by the virtue of equations of motions and boundary conditions for the steady flow. This equality gives us the variational principle: **any steady flow represents a stationary point on the isovortical sheet**. The only difference from the classical Arnold's result is the modified definition of the isovorticity condition.

$$E_{\theta\theta}\Big|_{\theta=0} = \int_{Q_0} \left(\mathbf{u}_\theta^2 + (\mathbf{W}\times\mathbf{f})\cdot\boldsymbol{\omega}_\theta\right)_{\theta=0} d\mathbf{x}$$

It shows that the stationary point of the energy functional in the 3D case always represents a saddle point.
Stability conditions for the steady plane flows: $W_1 = \partial\Psi/\partial x_2$, $W_2 = -\partial\Psi/\partial x_1$. The second variation is

$$E_{\theta\theta}\Big|_{\theta=0} = \int_{Q_0} \left(\mathbf{u}_\theta^2 - \frac{d\Psi}{d\Omega}\omega_\theta^2\right)_{\theta=0} dx_1 dx_2$$

where $\Psi = \Psi(\Omega)$ characterises the considered plane steady flow. Then, similar to the Arnold cases the inequalities with two constants C^-, C^+ and $C^- < -d\Psi/d\Omega < C^+$ give both sufficient linear and nonlinear stability conditions.

- We have introduced a class of fluid flow models, which is characterised by an additional advection with the drift velocity, which appears as an arbitrary given function. All these models have been obtained by regular asymptotic procedures.
- The drift velocity is not small, it is of the same order of magnitude as the averaged Eulerian velocity.
- These models include vortex dynamics, acoustics, and MHD; all they have important applications.

- The WVD was discovered by Craik and Leibovich in 1978 (CL-equation); they were focused on the description of Langmuir circulations generated by surface waves.
- Our main achievement is a drastic simplification of the derivation of WVD. The usual derivation of WVD equations is performed with the use of the GLM (by M. E. McIntyre). We introduce the WVD in its natural simplicity and generality. Our derivation is accessible to the 2nd year UG students.

- All considered models are Hamiltonian. Darryl Holm did it for the WVD in the GLM form, which is somehow different from ours. We leave the developing of the related Hamiltonian structures to the ‘Hamiltonian community’.
- The discussed analogy with stratification immediately leads to the Richardson type stability criteria ...
- A possibility of the finite-time singularity in the WVD vorticity field can be studied.
- Viscosity and/or density stratification can be routinely added to the WVD equations...

(2012) MHD drift equation: from Langmuir circulations to MHD dynamo? *J.Fluid Mech.* **698**, 51-61.

(2013) An asymptotic model in acoustics: Acoustic-drift equation. *J.Acoust.Soc.Am.*: 134 (5), 3419-3424.

(2013) On the self-propulsion velocity of an N-sphere micro-robot. *JFM Rapids*, 716, R1-1.

(2013) Dumbbell micro-robot driven by flow oscillation. *JFM Rapids*, 717, R8-1.

(2010) Admixture and drift in oscillating fluid flows. E-print: arXiv: 1009,4058v1

(2008) Viscous flows in a half space caused by tangential vibrations on its boundary. *Studies in Appl. Math.* **121**, 337

(2005) Vibrodynamics of pendulum and submerged solid. *J. Math. Fluid Mech.*, **7**, 397-412.

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Thanks

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