A Mixed Fluid-Kinetic Solver for the Vlasov-Poisson System

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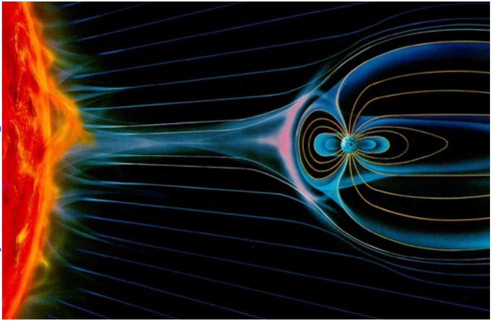
October 17th, 2014

Outline

Outline

Space weather modeling

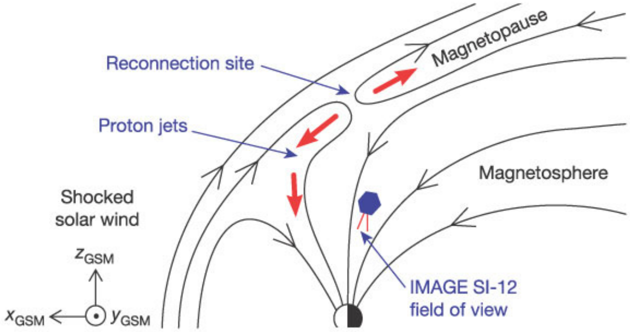
aerospaceweb.org



- Supersonic solar wind constantly bombarding Earth
- Solar wind $\equiv$ stream of energetic charged particles from Sun
- Earth's magnetic field $\Longrightarrow$ sets up magnetosphere, bow shock, . . .
- Solar flares can create geomagnetic storms, which can affect space satellites
- Challenge: accurately predicting space weather in real time

Collisionless magnetic reconnection

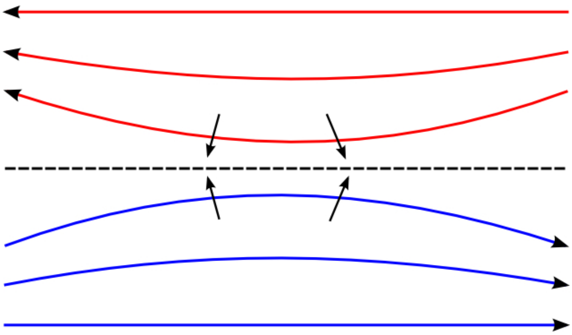
[Frey et al., Nature, 2003]



- Magnetic field lines from different magnetic domains are spliced to one another
- Creates rapid outflows away from reconnection point
- Outflows have important affect on space weather, can affect satellites, . . .
- Can happen both on the **dayside** as well as in the **magnetotail**

Collisionless magnetic reconnection

$B_1 = A_{,y}$ and $B_2 = -A_{,x}$



- **Starting point:** oppositely directed field lines are driven towards each other
- Field lines merge at the so-called **X-point**
- **Lower energy state:** change topology of field lines
- Results in large energy release in the form of oppositely directed jets

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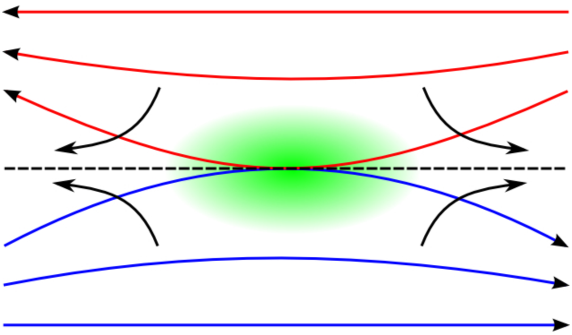

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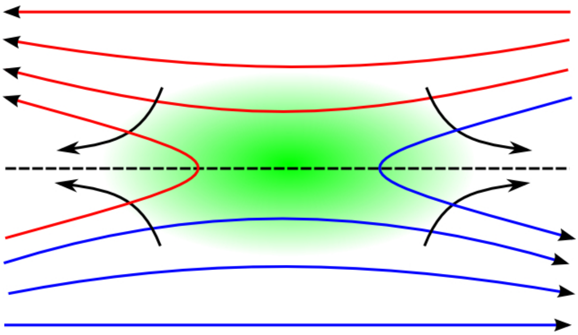

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Outline

Hierarchy of plasma models

Particles $\longrightarrow$ kinetic $\longrightarrow$ hybrid kinetic/fluid $\longrightarrow$ fluid

1. **Full particle description:** computationally intractable
2. **Kinetic description:**
   - Fully Lagrangian description via macro-particles
   - Particle-in-cell description
   - Semi-Lagrangian description
   - Eulerian description
3. **Hybrid description:** ion particles, electron fluid
4. **Fluid description:**
   - High-moment approximation (moment-closure)
   - 5-moment approximation (Euler equations)
   - Hall MHD (quasi-neutrality $\Longrightarrow$ single-fluid system)
   - MHD (ideal Ohm's law)

Mathematical models

Two species models: (1 ion, 1 electron)

- Vlasov-Maxwell model:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f_s = 0,$$

$$\frac{\partial}{\partial t} \begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix} + \nabla \times \begin{bmatrix} \mathbf{E} \\ -c^2 \mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -c^2 \mathbf{J} \end{bmatrix},$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = c^2 \sigma,$$

$$\sigma = \sum_s \frac{q_s}{m_s} \int f_s \, d\mathbf{v}, \quad \mathbf{J} = \sum_s \frac{q_s}{m_s} \int \mathbf{v} f_s \, d\mathbf{v}$$

- Two-fluid 10-moment model (Generalized Euler-Maxwell):

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \mathbb{E}_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2} \mathbf{v}\mathbf{v} \end{bmatrix} f_s \, d\mathbf{v} \qquad \left\{ \text{closure:} \quad \mathbb{Q} \equiv 0 \right\}$$

$$f_s(t, \mathbf{x}, \mathbf{v}) = \frac{\rho_s^{\frac{2+d}{2}}}{(2\pi)^{\frac{d}{2}} \sqrt{\det \mathbb{P}_s}} \exp\left[-\frac{\rho_s}{2} (\mathbf{v} - \mathbf{u}_s)^T \, \mathbb{P}_s^{-1} (\mathbf{v} - \mathbf{u}_s) \right]$$

Mathematical models

- (cont'd) Two-fluid 10-moment model

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho_s\\ \rho_s\mathbf{u}_s\\ \mathbb{E}_s\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho_s\mathbf{u}_s\\ \rho_s\mathbf{u}_s\mathbf{u}_s+\mathbb{P}_s\\ 3\,\mathrm{Sym}\left(\mathbf{u}_s\mathbb{E}_s\right)-2\rho_s\mathbf{u}_s\mathbf{u}_s\mathbf{u}_s\end{bmatrix} = \begin{bmatrix}0\\ \frac{q_s}{m_s}\rho_s\left(\mathbf{E}_s+\mathbf{u}_s\times\mathbf{B}\right)\\ 2\,\mathrm{Sym}\left(\frac{q_s}{m_s}\rho_s\mathbf{u}_s\mathbf{E}+\mathbb{E}_s\times\mathbf{B}\right)\end{bmatrix},$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} + \nabla\times\begin{bmatrix}\mathbf{E}\\ -c^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}0\\ -c^2\mathbf{J}\end{bmatrix},$$

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{E}=c^2\sigma,$$

$$\sigma=\sum_s\frac{q_s}{m_s}\rho_s,\quad \mathbf{J}=\sum_s\frac{q_s}{m_s}\rho_s\mathbf{u}_s$$

- Two-fluid 5-moment model (Euler-Maxwell):

$$\begin{bmatrix}\rho_s\\ \rho_s\mathbf{u}_s\\ \mathcal{E}_s\end{bmatrix} = \int\begin{bmatrix}1\\ \mathbf{v}\\ \frac{1}{2}\|\mathbf{v}\|^2\end{bmatrix} f_s\,d\mathbf{v} \qquad \left\{\text{closure: } \mathbb{P}\equiv\frac{1}{3}\mathrm{trace}(\mathbb{P})\mathbb{I}\right\},$$

$$f_s(t,\mathbf{x},\mathbf{v}) = \frac{\rho_s^{\frac{2+d}{2}}}{(2\pi p_s)^{\frac{d}{2}}}\exp\left[-\frac{\rho_s}{2p_s}\left(\mathbf{v}-\mathbf{u}_s\right)^T\left(\mathbf{v}-\mathbf{u}_s\right)\right]$$

Mathematical models

- (cont'd) Two-fluid 5-moment model

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho_s\\ \rho_s\mathbf{u}_s\\ \mathcal{E}_s\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho_s\mathbf{u}_s\\ \rho_s\mathbf{u}_s\mathbf{u}_s + p_s\mathbb{I}\\ \mathbf{u}_s\left(\mathcal{E}_s + p_s\right)\end{bmatrix} = \begin{bmatrix}0\\ \frac{q_s}{m_s}\rho_s\left(\mathbf{E}+\mathbf{u}_s\times\mathbf{B}\right)\\ \frac{q_s}{m_s}\rho_s\mathbf{u}_s\cdot\mathbf{E}\end{bmatrix},$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} + \nabla\times\begin{bmatrix}\mathbf{E}\\ -c^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}0\\ -c^2\mathbf{J}\end{bmatrix},$$

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\cdot\mathbf{E} = c^2\sigma,$$

$$\sigma = \sum_s \frac{q_s}{m_s}\rho_s, \quad \mathbf{J} = \sum_s \frac{q_s}{m_s}\rho_s\mathbf{u}_s$$

- MHD models

$$\text{Quasi-neutrality} \implies \rho = \rho_i + \rho_e, \quad \mathbf{u} = \frac{\rho_i\mathbf{u}_i + \rho_e\mathbf{u}_e}{\rho_i + \rho_e}, \quad p = p_i + p_e$$

$$c \to \infty \implies \nabla\times\mathbf{B} = \mathbf{J}$$

Mathematical models

- Generalized Ohm's law:

$$\mathbf{E} = \mathbf{B} \times \mathbf{u} \qquad \text{(Ohm's law)}$$
$$+\eta\,\mathbf{J} \qquad \text{(resistivity)}$$
$$+\left(\frac{m_i - m_e}{\rho}\right)\mathbf{J} \times \mathbf{B} \qquad \text{(Hall term)}$$
$$+\frac{1}{\rho}\nabla\left(m_e p_i - m_i p_e\right) \qquad \text{(pressure term)}$$
$$+\frac{m_i m_e}{\rho}\left\{\partial_t \mathbf{J} + \nabla\cdot\left(\mathbf{u}\mathbf{J} + \mathbf{J}\mathbf{u} + \frac{m_e - m_i}{\rho}\mathbf{J}\mathbf{J}\right)\right\} \qquad \text{(inertial term)}$$

- (cont'd) Resistive MHD model

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\mathbf{u} \\ \mathcal{E} \\ \mathbf{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \left(p + \frac{1}{2}\|\mathbf{B}\|^2\right)\mathbb{I} - \mathbf{B}\mathbf{B} \\ \mathbf{u}\left(\mathcal{E} + p + \frac{1}{2}\|\mathbf{B}\|^2\right) - \mathbf{B}\left(\mathbf{u}\cdot\mathbf{B}\right) \\ \mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}\end{bmatrix} = \begin{bmatrix}0 \\ \mathbf{0} \\ \eta\nabla\cdot\left[\mathbf{B}\times\left(\nabla\times\mathbf{B}\right)\right] \\ \eta\triangle\mathbf{B}\end{bmatrix}$$

$$\nabla\cdot\mathbf{B} = 0$$

Fast magnetic reconnection

GEM challenge problem

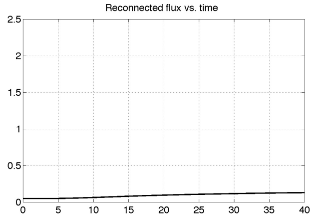
A brief history

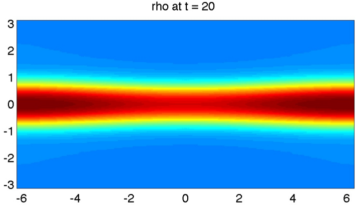
- Ideal MHD does not support magnetic reconnection
- Resistive MHD allows for **slow** magnetic reconnection
- **[Birn et al., 2001]:** Geospace Environment Modeling (GEM) challenge problem
- **[Shay et al., 2001]**: need $\frac{\partial \mathbf{J}}{\partial t}$, $\nabla \cdot \mathbb{P}$, or $\eta \mathbf{J}$ in Ohm's law to start
- Rate is independent of starting mechanism, important term is Hall: $\sim \mathbf{J} \times \mathbf{B}$
- **[Bessho and Bhattacharjee, 2007]**: in pair plasma important term is $\sim \nabla \cdot \mathbb{P}$
- **[Lazarian et al, 2012]**: fast reconnection in resistive MHD via turbulence

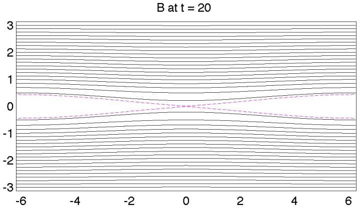
Reconnection rate vs. solution structure

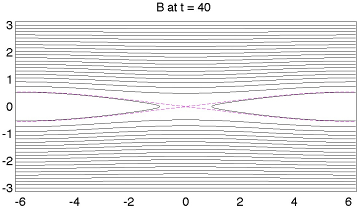
- Rate of magnetic reconnection is robust to many different models
- Hall MHD, various 2-fluid models, MHD with turbulence: all show similar rates
- Kinetic simulations show certain pressure tensor structure
- **Our goal:** higher moment models to match kinetic solution structures

GEM: Resistive MHD ($\eta = 5 \times 10^{-3}$)

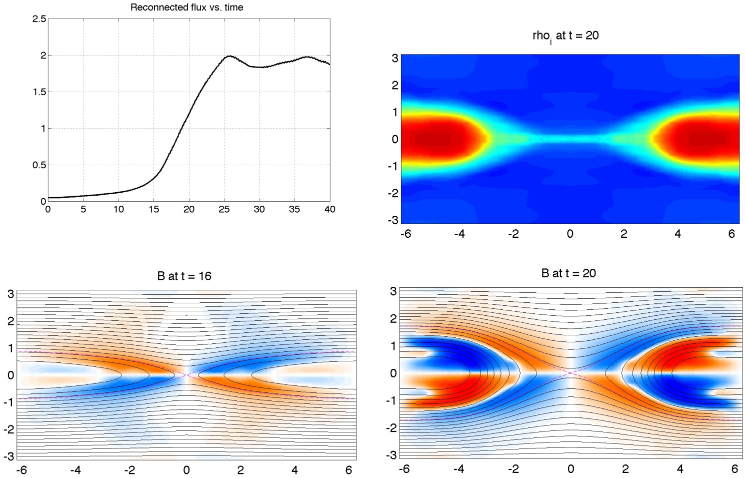








GEM: 2-fluid 5-moment ($m_i/m_e = 25$)



GEM: 10 and 20-moment with relaxation

[Johnson, 2011]

BGK Relaxation in higher-moment equations:

$$
\begin{aligned}
\rho_{,t} + \nabla\cdot(\rho\mathbf{u}) &= 0 \\
(\rho\mathbf{u})_{,t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbb{P}) &= 0 \\
\mathbb{E}_{,t} + \nabla\cdot(3\,\mathbf{u}\mathbb{E} - 2\rho\mathbf{u}\mathbf{u}\mathbf{u} + \mathbb{Q}) &= \frac{1}{\varepsilon}(p\mathbb{I} - \mathbb{P}) \\
\mathbb{F}_{,t} + \nabla\cdot\left(4\mathbf{u}\mathbb{F} - 6\mathbf{u}\mathbf{u}\mathbb{E} + 3\rho\mathbf{u}\mathbf{u}\mathbf{u}\mathbf{u} + \frac{3\mathbb{P}\mathbb{P}}{\rho}\right) &= -\frac{1}{\varepsilon}\mathbb{Q}
\end{aligned}
$$

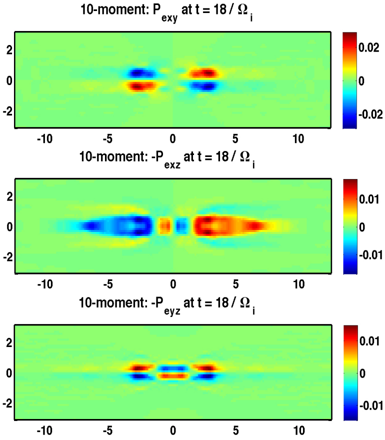
- **Chapman-Enskog expansion:**

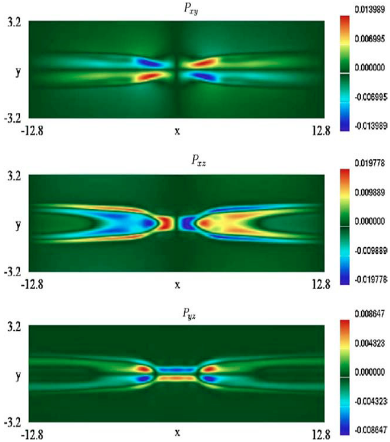
$$(\rho\mathbf{u})_{,t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + p\mathbb{I}) = \varepsilon\nabla^2\mathbf{u} + \mathcal{O}\left(\varepsilon^2\right)$$

- 10-moment with relaxation: we now have physical viscosity, not just numerical
- For a range of ε: $0 < \varepsilon \ll 1$, we get fast reconnection
- Furthermore, we can reproduce off-diagonal pressure from kinetic models
- 20-moment with relaxation: we now have non-zero heat flux

GEM: 10-moment with relaxation ($m_i/m_e = 25$)

[Johnson, 2011]

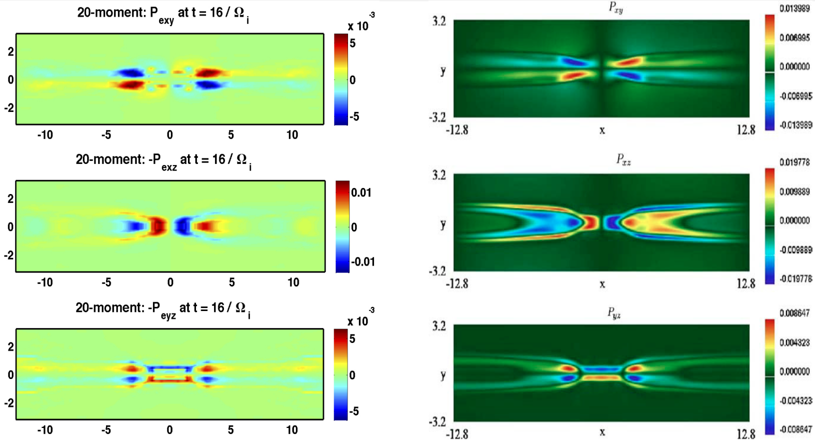




- **Conclusion:** qualitative agreement
- **Missing ingredient:** heat flux $\Longrightarrow$ need to go to higher moment models

GEM: 20-moment with relaxation ($m_i/m_e = 25$)

[Johnson, in prep]



- **Conclusion:** better qualitative agreement
- **Missing ingredient:** non-zero kurtosis: $\mathbb{K} = \mathbb{R} - \frac{3\mathbb{PP}}{\rho}$

Outline

Moments vs. Multiphysics

Summary of higher-moment approach:

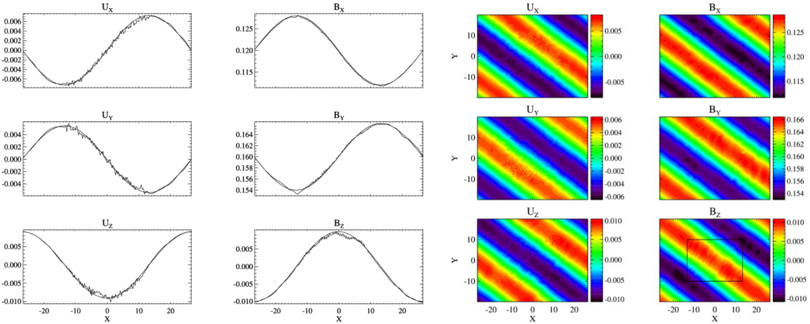
- Can get good qualitative agreement on GEM challenge problem
- Need to artificially introduce collisions (kinetic system is collisionless)
- Simulations are challenging due to density and pressure positivity violations
- May need very large number of moments in very rarefied regimes
- Other micro-scale phenomena may not be well-captured (open problem)

Multiphysics approach (i.e., domain decomposition):

- Use low-moment fluid solver where possible
- Use kinetic solver where necessary
- **Challenge #1:** how to communicate between different solvers
- **Challenge #2:** how to adaptively choose regions (a posteriori error estimates)
- Many options for models, coupling mechanisms, numerical methods, . . .

State-of-the-art: Hall MHD + IPIC

[Daldorff, Tóth, Gombosi, Lapenta, Amaya, Markidis, & Brackbill 2014]



- Whistler wave example: Implicit PIC code region embedded in Hall MHD model
- Restriction (PIC $\mapsto$ Hall MHD): modified Ohm's law
- Prolongation (Hall MHD $\mapsto$ PIC): boundary conditions of PIC region
- **Disadvantage #1:** consistency problems between models (quasineutrality)
- **Disadvantage #2:** PIC introduces statistical noise

Outline

Vlasov-Poisson system

Electrostatic approximation

- Some simplifying assumptions:
  1. Two species: ions $(+)$ & electrons $(-)$
  2. Slow moving charges $\Longrightarrow$ electrostatics
  3. Track electrons, assume fixed background ions
- Electrons are described by a probability density function:

$$f(t,\mathbf{x},\mathbf{v}) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

- Moments of $f(t,\mathbf{x},\mathbf{v})$ correspond to various physical observables:

$$\rho(t,\mathbf{x}) := \int f \, d\mathbf{v}, \quad \rho\mathbf{u}(t,\mathbf{x}) := \int \mathbf{v} \, f \, d\mathbf{v}, \quad \mathcal{E}(t,\mathbf{x}) := \frac{1}{2}\int \|\mathbf{v}\|^2 f \, d\mathbf{v}$$

- The Vlasov-Poisson system:

$$f_{,t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f - \mathbf{E}\cdot\nabla_{\mathbf{v}} f = 0,$$

$$\mathbf{E} = -\nabla_{\mathbf{x}}\phi, \quad -\nabla_{\mathbf{x}}^2\phi = \rho_0 - \rho(t,\mathbf{x})$$

Vlasov-Poisson system

Properties

- **Characteristics (Vlasov is an advection equation in phase space):**

$$\left(\mathbf{X}\left(t;\mathbf{x},\mathbf{v},s\right),\mathbf{V}\left(t;\mathbf{x},\mathbf{v},s\right)\right) \implies \frac{d\mathbf{X}}{dt}=\mathbf{V}(t), \qquad \frac{d\mathbf{V}}{dt}=-\mathbf{E}\left(t,\mathbf{X}(t)\right),$$

$$f\left(t,\mathbf{x},\mathbf{v}\right)=f_0\left(\mathbf{X}\left(0;t,\mathbf{x},\mathbf{v}\right),\mathbf{V}\left(0;t,\mathbf{x},\mathbf{v}\right)\right)$$

- **Maximum principle:**

$$0 \le \min_{(\mathbf{x},\mathbf{v})} f_0\left(\mathbf{x},\mathbf{v}\right) \le f\left(t,\mathbf{x},\mathbf{v}\right) \le \max_{(\mathbf{x},\mathbf{v})} f_0\left(\mathbf{x},\mathbf{v}\right)$$

- **Conserved functional:**

$$\frac{d}{dt}\int_{\mathbf{x}}\int_{\mathbf{v}} G(f)\,d\mathbf{v}\,d\mathbf{x}=0 \implies L_p \text{ norm: } \; G(f)=|f|^p, \quad \text{entropy: } \; G(f)=-f\ln f$$

- **Conservation laws:**

Mass: $$\rho_{,t}+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u})=0, \qquad \frac{d}{dt}\int_{\mathbf{x}}\rho\,d\mathbf{x}=0$$

Momentum: $$(\rho\mathbf{u})_{,t}+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\mathbf{u}+\mathbb{P})=-\rho\mathbf{E}, \qquad \frac{d}{dt}\int_{\mathbf{x}}\rho\mathbf{u}\,d\mathbf{x}=0$$

Total energy: $$\left(\mathcal{E}+\frac{1}{2}\|\mathbf{E}\|^2\right)_{,t}+\nabla_{\mathbf{x}}\cdot\mathcal{F}=0, \qquad \frac{d}{dt}\int_{\mathbf{x}}\left(\mathcal{E}+\frac{1}{2}\|\mathbf{E}\|^2\right)d\mathbf{x}=0$$

Outline

Outline

Mixed Fluid-Kinetic Solver

1+1 Vlasov-Poisson:

$$f_{,t} + vf_{,x} - Ef_{,v} = 0, \quad E_{,x} = \rho_0 - \rho(t,x)$$

Kinetic solver

- Operator-split semi-Lagrangian DG scheme (`dogpack-code.org`)
- In current experiments: global kinetic solver
- Work in progress: local kinetic solver

Fluid solver

- Standard RKDG scheme (`dogpack-code.org`)
- Solve the “20”-moment model of **[Groth, Gombosi, Roe, & Brown, 1994, 2003]**

Coupling

- Kinetic $\mapsto$ fluid: correct moment-closure
- Fluid $\mapsto$ kinetic: quadrature-based moment-closure reconstructions
- Why couple fluid back to kinetic? keep model consistency

Gaussian-based moment closure
[Groth, Gombosi, Roe, & Brown, 1994, 2003]

- Knock-out missing moments by pretending they come from a Gaussian

$$\mathbf{20-moment}: \quad \mathbb{R} \leftarrow 3\mathbb{P}\mathbb{P}/\rho \quad (\text{Kurtosis: } \mathbb{K} = \mathbb{R} - 3\mathbb{P}\mathbb{P}/\rho \equiv 0)$$

$$\mathbf{35-moment}: \quad \mathbb{S} \leftarrow 10\rho\mathbb{P}\mathbb{Q}$$

- Example: 20-moments in 1D (reduces to only 4 moments):

$$q = \begin{bmatrix} \rho \\ \rho u \\ p + \rho u^2 \\ q + 3pu + \rho u^3 \end{bmatrix} \quad \text{and} \quad f(q) = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ q + 3pu + \rho u^3 \\ \frac{3p^2}{\rho} + 4qu + 6pu^2 + \rho u^4 + (\mathbb{K} = 0) \end{bmatrix}$$

- Four eigenvalues of flux Jacobian:

$$\lambda = u + s\sqrt{\frac{p}{\rho}}, \qquad s^4 - 6s^2 - 4sh + 3 = 0, \qquad h := \frac{q}{p}\sqrt{\frac{\rho}{p}}$$

- **Advantage:** no direct moment inversion
- **Disadvantage:** limited hyperbolicity: $|h| < \sqrt{\sqrt{8} - 2} \approx 0.9102$

Outline

1D quadrature-based moment-closure

Basic idea **[Rodney Fox (ISU) et al]**

- Assume that distribution is a finite sum of Dirac-deltas:

$$f(t,x,v) = \sum_{k=1}^{N} \omega_k(t,x)\,\delta(v - \mu_k(t,x))$$

- **NOTE:** This is reminiscent of PIC, except physical space is left continuous
- **NOTE:** Similar to discrete vel models, except each x has different velocities
- To find weights and abscissas, match first $2N$ moments:

$$\int_{-\infty}^{\infty} f\,dv = M_0 = \sum_{k=1}^{N} \omega_k, \quad \int_{-\infty}^{\infty} v\,f\,dv = M_1 = \sum_{k=1}^{N} \omega_k\,\mu_k, \quad \dots$$

$$\int_{-\infty}^{\infty} v^{2N-1}\,f\,dv = M_{2N-1} = \sum_{k=1}^{N} \omega_k\,\mu_k^{2N-1}$$

- **Closure:**

$$M_{2N} = \sum_{k=1}^{N} \omega_k\,\mu_k^{2N}$$

- Using GQ can reformulate this as a root finding problem for an N^{th} degree poly

1D quadrature-based moment-closure

Basic idea **[Rodney Fox (ISU) et al]**

Quadrature points & weights:

$$\int_{-\infty}^{\infty} g(v)\, w(v)\, dv \approx \sum_{k=1}^{N} \omega_k\, g(\mu_k)$$

- Weight function satisfies

$$\int_{-\infty}^{\infty} v^k\, w(v)\, dv = M_k \quad \text{for} \quad k = 0, 1, 2, 3, \ldots$$

- If we make exact for $g(v) = 1,\ v,\ v^2,\ \ldots$ we arrive at moment-closure eqns
- Can solve these equations by constructing orthogonal polynomials:

$$\langle g, h \rangle_w := \int_{-\infty}^{\infty} g(v)\, h(v)\, w(v)\, dv$$

- e.g., up to second order:

$$\psi^{(0)}(v) = 1, \qquad \psi^{(1)}(v) = v - u,$$
$$\psi^{(2)}(v) = 3\rho p v^2 - (6\rho p u + 3\rho q)\, v + \left(3\rho p u^2 - 3p^2 + 3u\rho q\right)$$

1D quadrature-based moment-closure

M^4 closure ($N = 2$)

M^4 closure model:

$$\begin{bmatrix} \rho \\ \rho u \\ \rho u^2 + p \\ \rho u^3 + 3pu + q \end{bmatrix}_{,t} + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u^3 + 3pu + q \\ \rho u^4 + 6pu^2 + 4uq + \frac{q^2}{p} + \frac{p^2}{\rho} \end{bmatrix}_{,x} = 0$$

Hyperbolic structure:

- Eigenvalues (each has algebraic multiplicity 2, geometric multiplicity 1):

$$\mu_1 = \lambda^{(1)} = \lambda^{(2)} = u + \frac{q}{2p} - \sqrt{\frac{p}{\rho} + \left(\frac{q}{2p}\right)^2}$$

$$\mu_2 = \lambda^{(3)} = \lambda^{(4)} = u + \frac{q}{2p} + \sqrt{\frac{p}{\rho} + \left(\frac{q}{2p}\right)^2}$$

- Weak hyperbolicity with 2 linearly degenerate waves
- Delta shocks form for generic initial data

1D quadrature-based moment-closure

Bi-Gaussian ansatz **[Chalons, Fox, & Massot, 2010]**

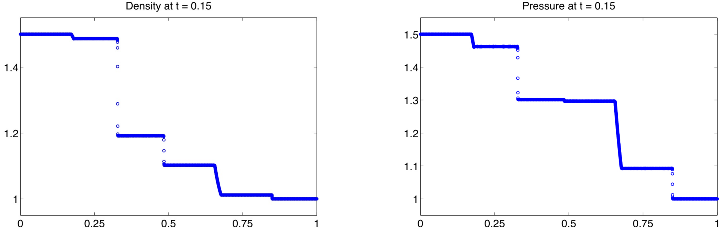
$$f(t,x,v) = \frac{\omega_1}{\sqrt{2\pi\sigma}} e^{-\frac{(v-\mu_1)^2}{2\sigma}} + \frac{\omega_2}{\sqrt{2\pi\sigma}} e^{-\frac{(v-\mu_2)^2}{2\sigma}}$$

- Moment-closure equations ($\rho\sigma = p(1-\alpha)$):

$$\omega_1\mu_1^0 + \omega_2\mu_2^0 = \rho, \quad \omega_1\mu_1^1 + \omega_2\mu_2^1 = \rho u, \quad \omega_1\mu_1^2 + \omega_2\mu_2^2 = \rho u^2 + \alpha p,$$

$$\omega_1\mu_1^3 + \omega_2\mu_2^3 = \rho u^3 + 3\alpha p u + q, \quad \omega_1\mu_1^4 + \omega_2\mu_2^4 = \rho u^4 + 6\alpha p u^2 + 4 q u + r + \frac{3p^2(\alpha^2-1)}{\rho},$$

- Riemann solution:



Moment-realizability condition

Bi-Gaussian distribution **[Chalons, Fox, & Massot, 2010]**

Theorem (Moment-realizability condition for the bi-Gaussian distribution)

Assume that the primitive variables satisfy the following conditions:

$$0 < \rho, \quad 0 < p, \quad \frac{p^3 + \rho q^2}{\rho p} \leq r, \quad \text{If } q = 0\text{:} \ \frac{p^2}{\rho} \leq r \leq \frac{3p^2}{\rho}$$

1. *If* $q \neq 0$ *then* $\exists!\ \alpha \in (0,1]$ *that satisfies the following cubic polynomial:*

   $$\mathcal{P}(\alpha) = 2p^3\alpha^3 + \left(\rho r - 3p^2\right) p\alpha - \rho q^2 = 0.$$

   From this α *we can uniquely obtain the quadrature abscissas and weights.*

2. *If* $q = 0$ *and* $\frac{p^2}{\rho} \leq r < \frac{3p^2}{\rho}$ *then* $\exists!\ \alpha \in (0,1]$ *such that* $\alpha = \sqrt{\frac{3p^2 - \rho r}{2p^2}}$. *The quadrature abscissas and weights are again unique.*

3. *If* $q = 0$ *and* $r = \frac{3p^2}{\rho}$, *then* $\alpha = 0$. *This case corresponds to a single Gaussian distribution. In this case we lose uniqueness of the quadrature abscissas and weights.*

1D quadrature-based moment-closure

Bi-B-spline ansatz **[Cheng and R, 2013]**

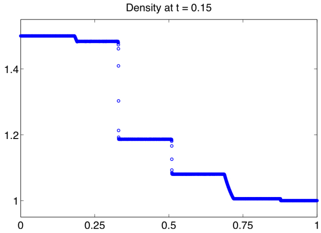
$$f(t,x,v) = \omega_1 B^0_\sigma(v-\mu_1) + \omega_2 B^0_\sigma(v-\mu_2), \quad B^0_\sigma(v) = \begin{cases} \frac{2}{\sigma}\left(2v+\sqrt{\sigma}\right) & \text{if } -\sqrt{\sigma} \le 2v \le 0 \\ \frac{2}{\sigma}\left(\sqrt{\sigma}-2v\right) & \text{if } 0 \le 2v \le \sqrt{\sigma} \end{cases}$$

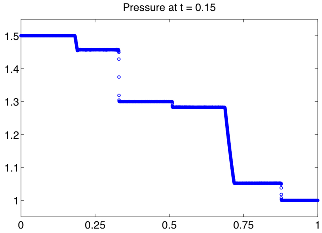
- Moment-closure equations ($\rho\sigma = 24p(1-\alpha)$):

$$\omega_1\mu_1^0 + \omega_2\mu_2^0 = \rho, \quad \omega_1\mu_1^1 + \omega_2\mu_2^1 = \rho u, \quad \omega_1\mu_1^2 + \omega_2\mu_2^2 = \rho u^2 + \alpha p,$$

$$\omega_1\mu_1^3 + \omega_2\mu_2^3 = \rho u^3 + 3\alpha p u + q, \quad \omega_1\mu_1^4 + \omega_2\mu_2^4 = \rho u^4 + 4qu + 6\alpha p u^2 + r + \frac{6p^2}{5\rho}(3\alpha+2)(\alpha-1)$$

- Riemann solution:

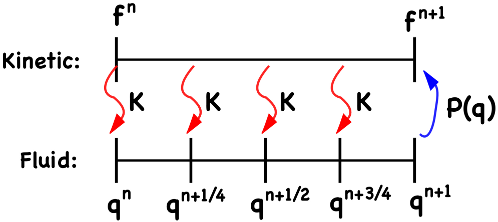




Outline

Kinetic model $\mapsto$ fluid model

Restriction



Strategy:

- Evolve the kinetic equation from $t = t^n$ to $t = t^n + \Delta t$
- From kinetic solution create kurtosis interpolant on $[t^n, t^n + \Delta t]$:

$$\text{e.g.,} \quad \tilde{\mathbb{K}}(t,x)\Big|_{\mathcal{T}_i \times [t^n, t^{n+1}]} := \frac{(t^{n+1} - t)}{\Delta t} \sum_{\ell=1}^{M} \tilde{\mathbb{K}}_i^{n(\ell)} \varphi^{(\ell)} + \frac{(t - t^n)}{\Delta t} \sum_{\ell=1}^{M} \tilde{\mathbb{K}}_i^{n+1(\ell)} \varphi^{(\ell)}$$

- Solve corrected (**restriction**) “20-moment” fluid eqn from $t = t^n$ to $t = t^n + \Delta t$
- Correct kinetic soln to match first few moments of fluid soln (**prolongation**)

Fluid model $\mapsto$ kinetic model

Prolongation

- At $t = t^{n+1}$ compute from $f(t^{n+1}, x, v)$:

$$\tilde{M}_0, \quad \tilde{M}_1, \quad \tilde{M}_2, \quad \tilde{M}_3, \quad \tilde{M}_4$$

- Compute a reconstruction of this data (using **bi-B-spline** fluid moment closure)

$$g^{n+1}(x,v) := \mathcal{F}\left(v; \tilde{M}_0, \ldots, \tilde{M}_4\right)$$
$$\Delta f^{n+1}(x,v) := f\left(t^{n+1}, x, v\right) - g^{n+1}(x,v)$$

- At $t = t^{n+1}$ compute from fluid model:

$$M_0, \quad M_1, \quad M_2, \quad M_3, \quad M_4$$

- Compute a reconstruction of this data (using **bi-B-spline** fluid moment closure)

$$h^{n+1}(x,v) := \mathcal{F}\left(v; M_0, \ldots, M_4\right)$$

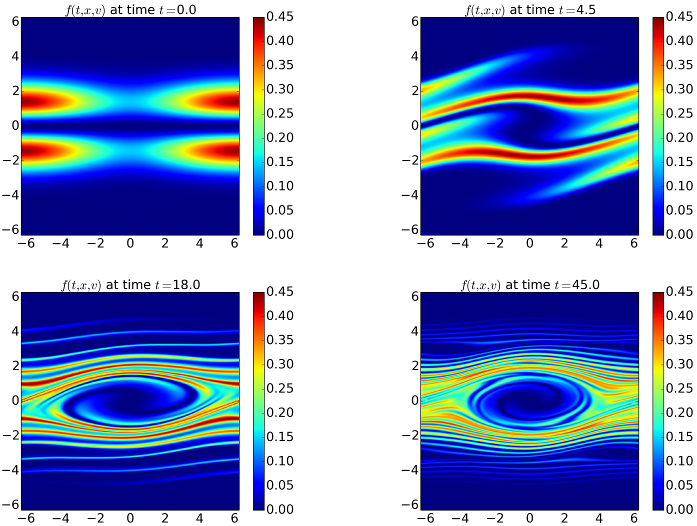
- Replace $g^{n+1}(x,v)$ by $h^{n+1}(x,v)$:

$$f\left(t^{n+1}, x, v\right) \leftarrow h^{n+1}(x,v) + \Delta f^{n+1}(x,v)$$

Outline

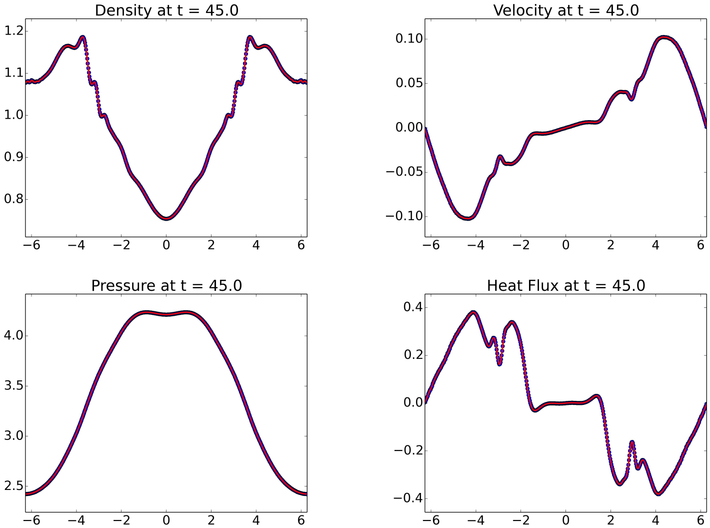
Two-stream instability

Bi-B-spline prolongation



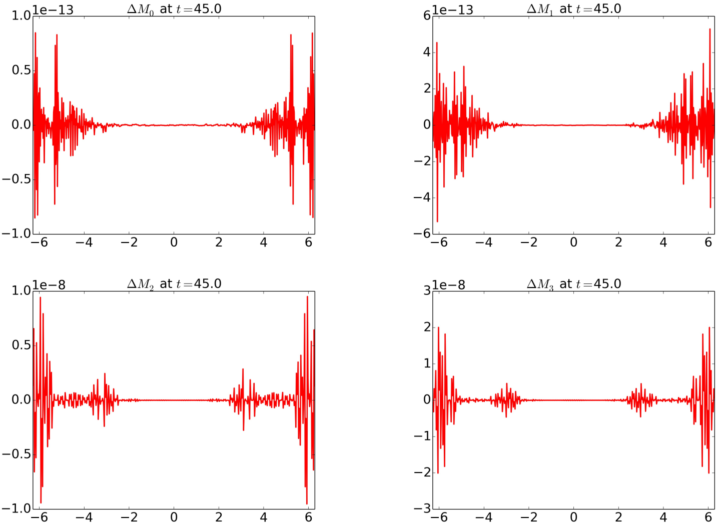
Two-stream instability

Bi-B-spline prolongation (red: fluid, blue: kinetic)



Two-stream instability

Bi-B-spline prolongation



Outline

Conclusions & future work

Multi-moment fluid models for GEM challenge

- 5-moment: correct reconnection rate, relies on numerical diffusion
- 10- and 20-moment: qualitatively correct pressure tensor
- 10- and 20-moment: need relaxation terms to get physically correct results
- Want to explore multi-physics (i.e., domain decomposition) approaches

Quadrature-Based Moment-Closure Models

- Moment-closure problem: assume a distribution, moment inversion
- Quadrature-based moment-closure allows for non-zero heat flux
- Quadrature via Dirac delta, Gaussians, B-splines

Mixed fluid/kinetic solvers (multiscale solvers)

- Restriction: Kinetic-to-fluid mapping via temporal interpolation
- Prolongation: Fluid-to-kinetic via reconstruction using moment-closures
- **Future work:** Problems where kinetic solver is not needed globally